# Proving theorems with quantifier elimination techniques

Stéphane Graham-Lengrand

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# Quantifier elimination

A property that is very simple to express:

A theory $\mathcal{T}$ admits quantifier elimination if for every formula $A$ there is a quantifier-free formula $B$ such that $\mathcal{T} \models A \Leftrightarrow B$ (with $\mathrm{fv}(B) \subseteq \mathrm{fv}(A)$)

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The property is useful for showing the decidability of provability/satisfiability in $\mathcal{T}$:
If

1. it can be shown constructively that $\mathcal{T}$ admits quantifier elimination (there is an algorithm that produces $B$ from $A$),
2. and provability/satisfiability in $\mathcal{T}$ of closed quantifier-free formulae is decidable,

then provability/satisfiability in $\mathcal{T}$ of all formulae is decidable.

# Examples

- Linear arithmetic over natural numbers / integers
  Presburger, 1929
- Linear arithmetic over rational / real numbers
  Fourier, 1827 - Motzkin, 1936
- Non-linear arithmetic over real / complex numbers
  Tarski, 1948
- Abelian groups — Szmielew, 1955
- Absolutely free term algebras — Malcev, 1971

# The interesting quantified form

$\mathcal{T}$ admits quantifier elimination as soon as formulae of the form

$$\exists y(l_1 \wedge \cdots \wedge l_m)$$

(where $l_1, \ldots, l_n$ are literals with free variables in $\{x_1, \ldots, x_n, y\}$) can be transformed into an equivalent quantifier-free form $B$ (with free variables in $\{x_1, \ldots, x_n\}$).

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Indeed, quantifiers can then be eliminated from any formula $A$, by induction on $A$:

- $\exists yA$:
  - eliminate quantifiers from $A$ to get $\exists yB$ with quantifier-free $B$,
  - turn $B$ into DNF,
  - distribute $\exists y$ over disjunctions
    to get a disjunction of formulae of the form $\exists y(l_1 \wedge \cdots \wedge l_m)$
  - apply the main hypothesis;
- $\forall yA$: express it first as $\neg\exists y\neg A$, then eliminate quantifiers from $\neg A$ and continue as above;
- propositional connectives: straightforward

# Semantical interpretation of the transformation

Writing $\overrightarrow{x}$ for $x_1, \ldots, x_n$, the transformation

$$\exists y(l_1(\overrightarrow{x}, y) \wedge \cdots \wedge l_m(\overrightarrow{x}, y)) \rightsquigarrow B(\overrightarrow{x})$$

can be interpreted semantically as follows:
In a $\mathcal{T}$-model $\mathfrak{M}$, the set of $n+1$-tuples satisfying $\mathfrak{M}(l_1 \wedge \cdots \wedge l_m)$ is projected as a set of $n$-tuples satisfying $\mathfrak{M}(B)$.

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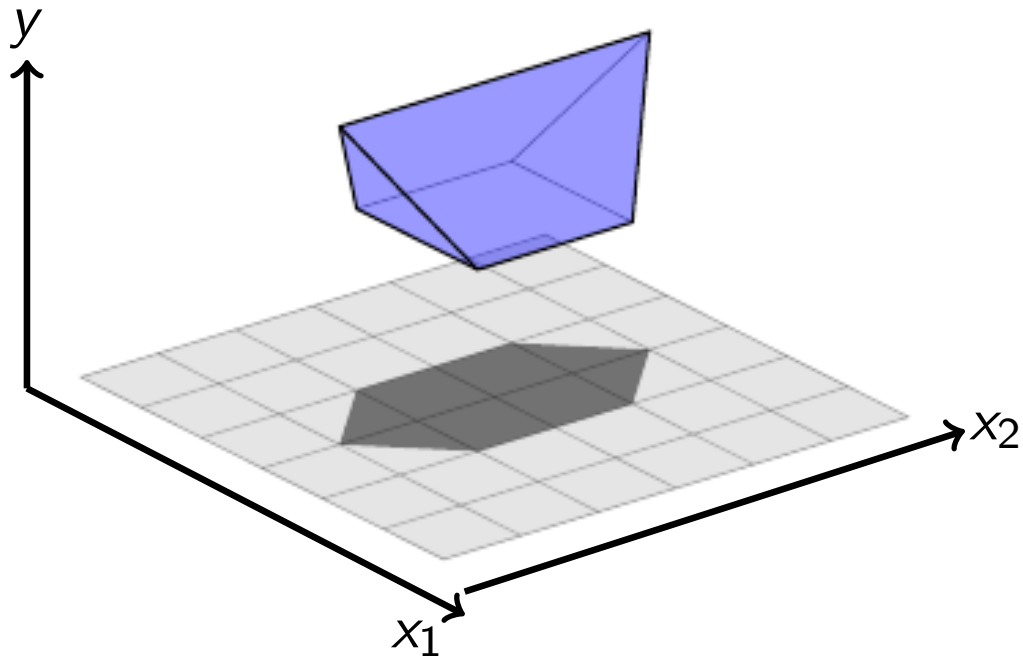
Example in linear arithmetic:



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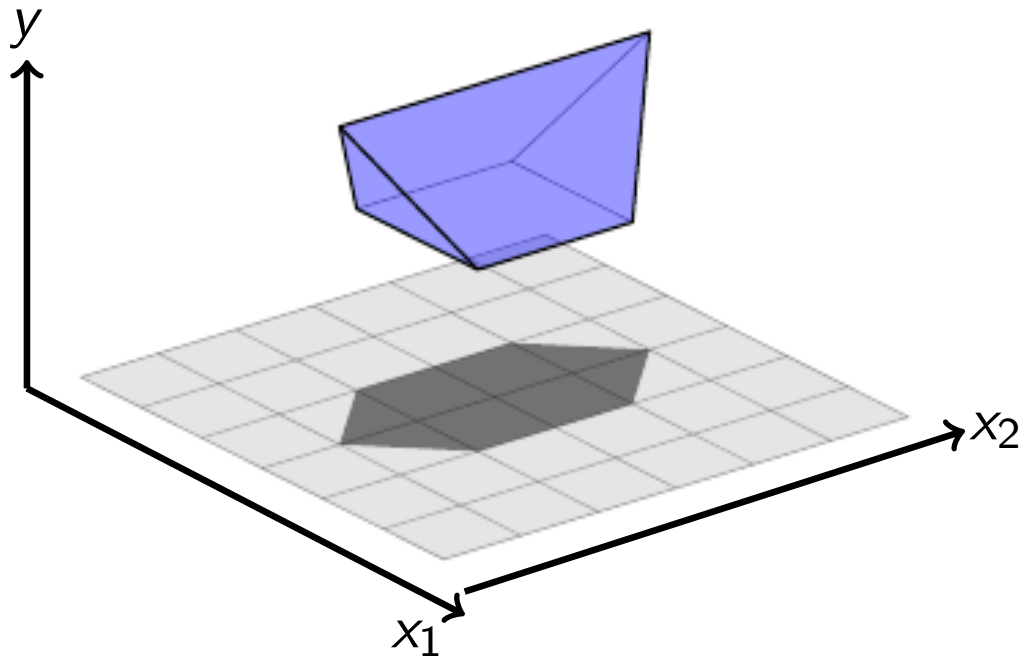


Similar interpretation in non-linear arithmetic, just with more complex shapes

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where $A_p$ is a quantifier-free formula with $\mathcal{T} \models A_p$,
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(of course in powerful logics we could use reflexivity and a correctness proof of the algorithm)

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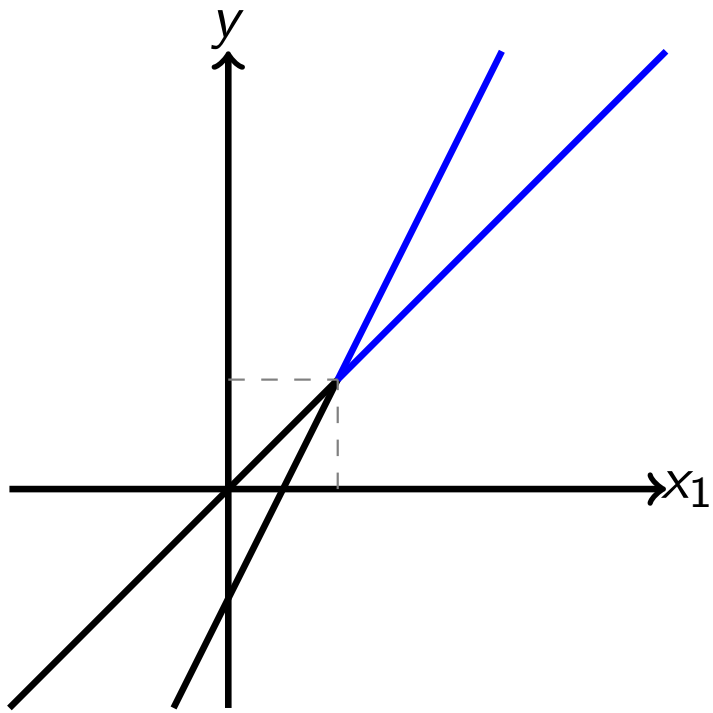

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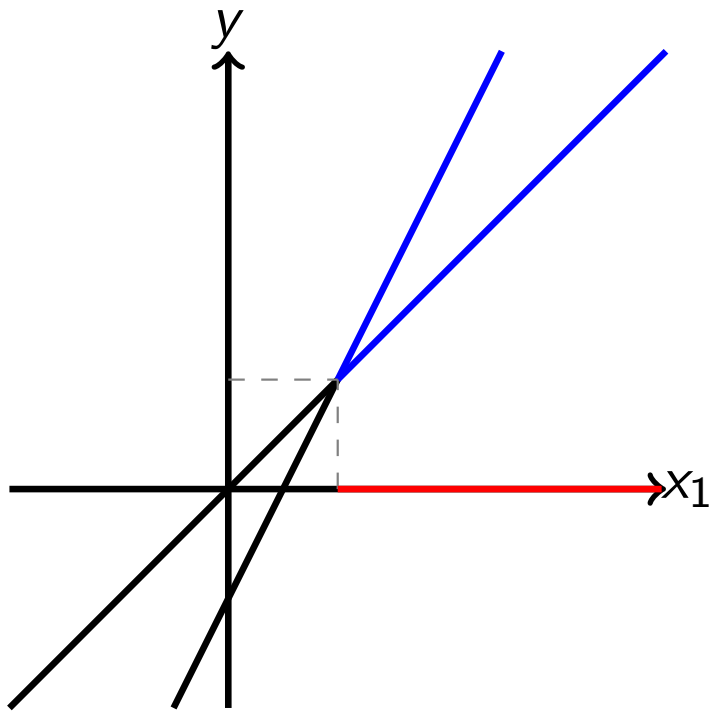

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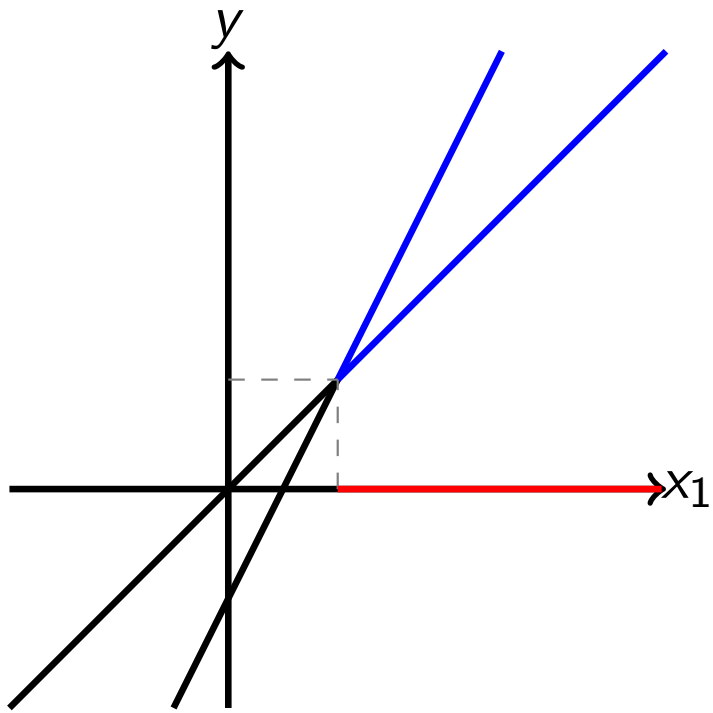

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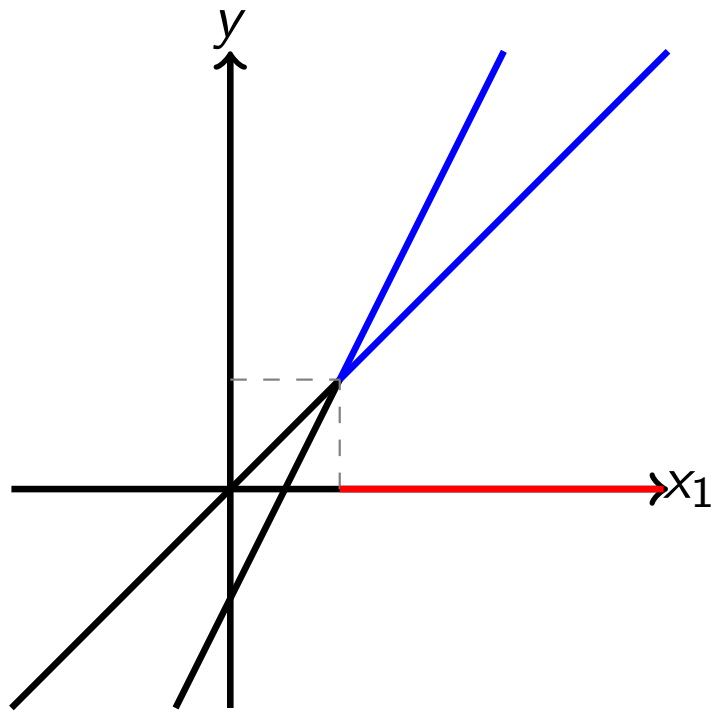


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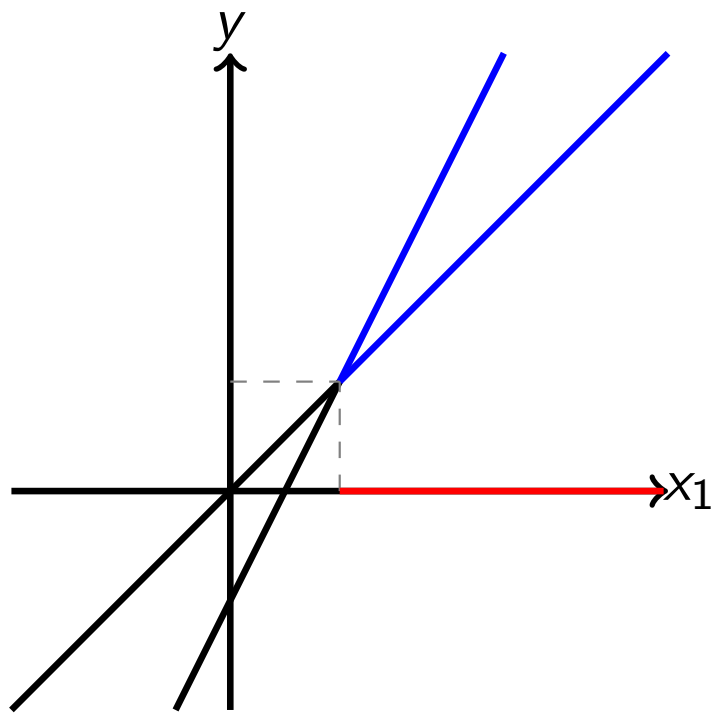


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with the premisses being normalised in $y$,
and $\lessdot_i \in \{\leq, <\}$ such that...

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Eliminate $x_n$, then $x_{n-1}$, etc, then $x_1$, then see if you have $\top$ or $\bot$

This is one of the ways to prove that Boolean resolution is refutationally complete in propositional logic:

$$\frac{C_1 \vee \neg y \quad C_2 \vee y}{C_1 \vee C_2}$$

If $B$ is a propositional CNF (a set of clauses), apply all possible resolutions on $x_n$, keep the new clauses, forget the clauses with $x_n$. Then do the same with $x_{n-1}$, etc. At the end: the set of clauses is $\emptyset$ ($B$ is sat) or $\{\bot\}$ ($B$ is unsat). In the latter case, this quantifier elimination process produces a proof of $B \Rightarrow \bot$ as a resolution proof.

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Quantifier elimination is one strategy for applying resolutions until $B$ can be determined to be sat or unsat.

But we know there's a better one: Conflict-Driven Clause Learning (CDCL), the main algorithm used in SAT-solvers.

- resolutions are performed "lazily"
- in response to explicit model construction attempts for $B$ that "failed" (the truth values that have been attempted are in conflict with one of the clauses)

# Back to LRA

Boolean resolution

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Could it be generalised to other theories with similar QE mechanisms?

# A lazy approach to quantifier elimination

CDCL (Conflict-Driven Clause Learning)

- procedure for deciding the satisfiability of Boolean formulae
- uses assignments of Boolean values to variables, e.g., $l \leftarrow \text{true}$

MCSAT (Model-Constructing Satisfiability) [dMJ13, Jov17]

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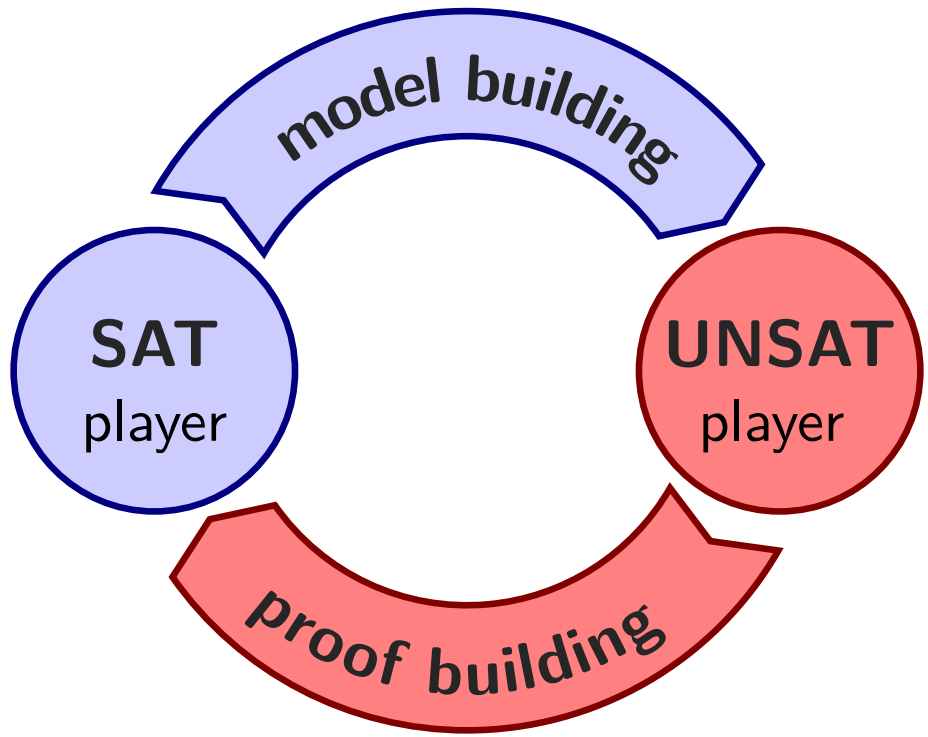
CDSAT (Conflict-Driven Satisfiability) [BGLS17, BGLS17]

- generalises MCSAT: generic combinations of abstract theories
- can also use first-order assignments
- models theory reasoning with modules made of inference rules

# Conflict-driven reasoning

2-player game to determine whether a formula is satisfiable.
It involves a trail where a putative model is being specified.
It relies on a notion of conflict between the putative model and the formula it should satisfy.



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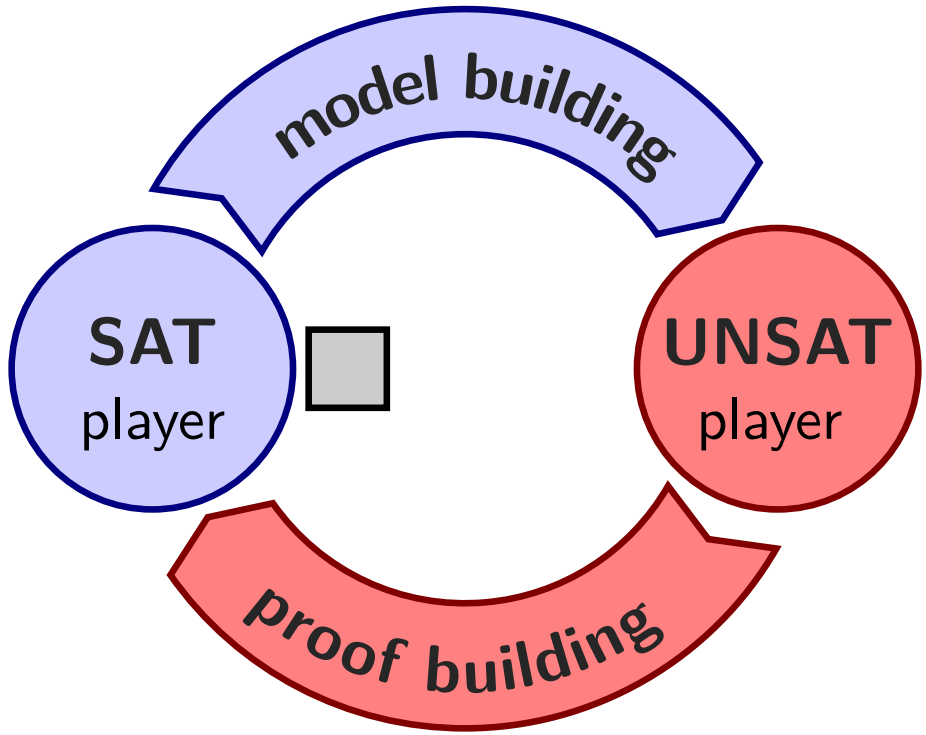

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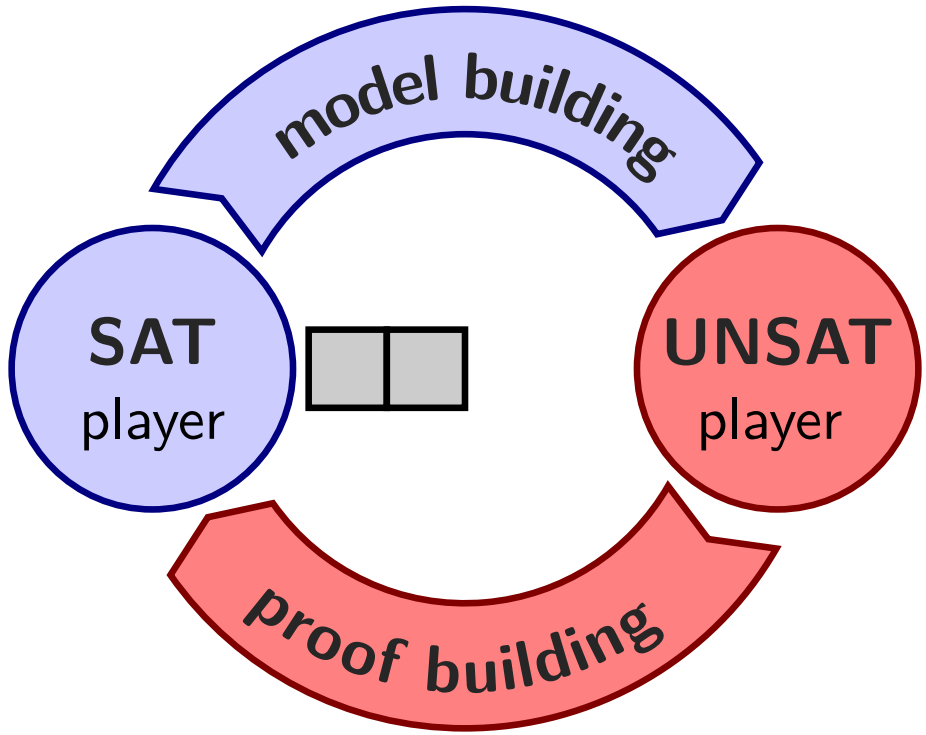

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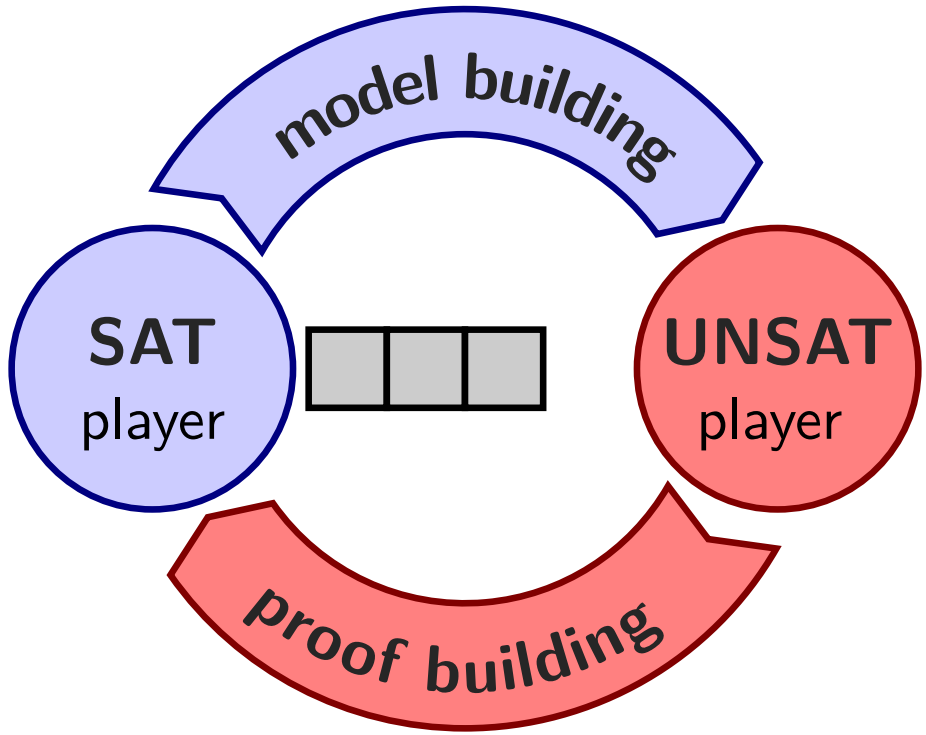

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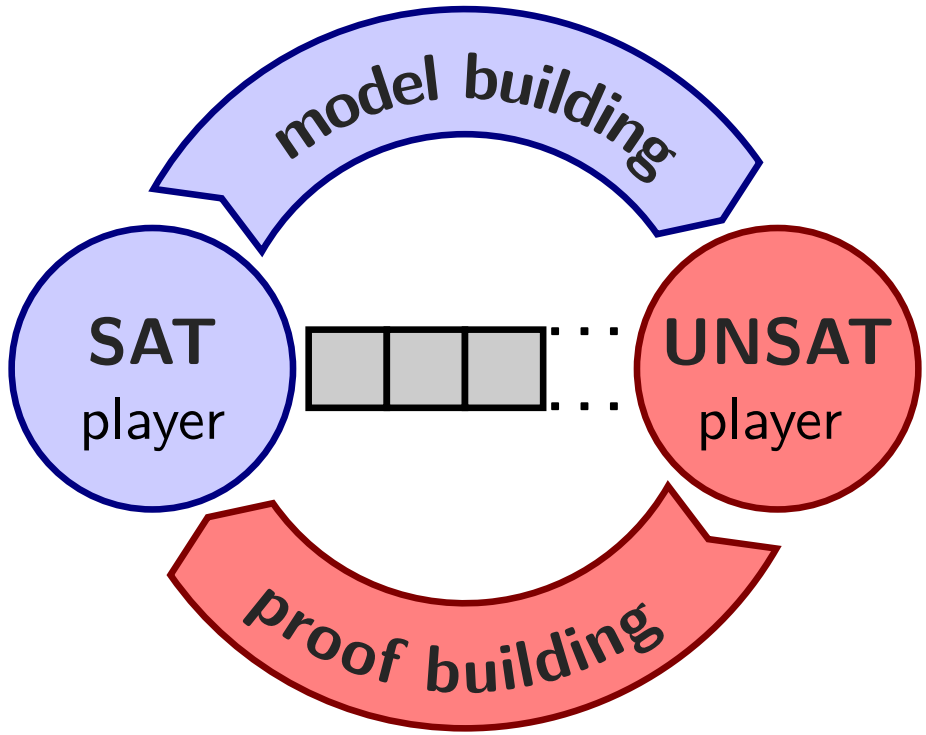

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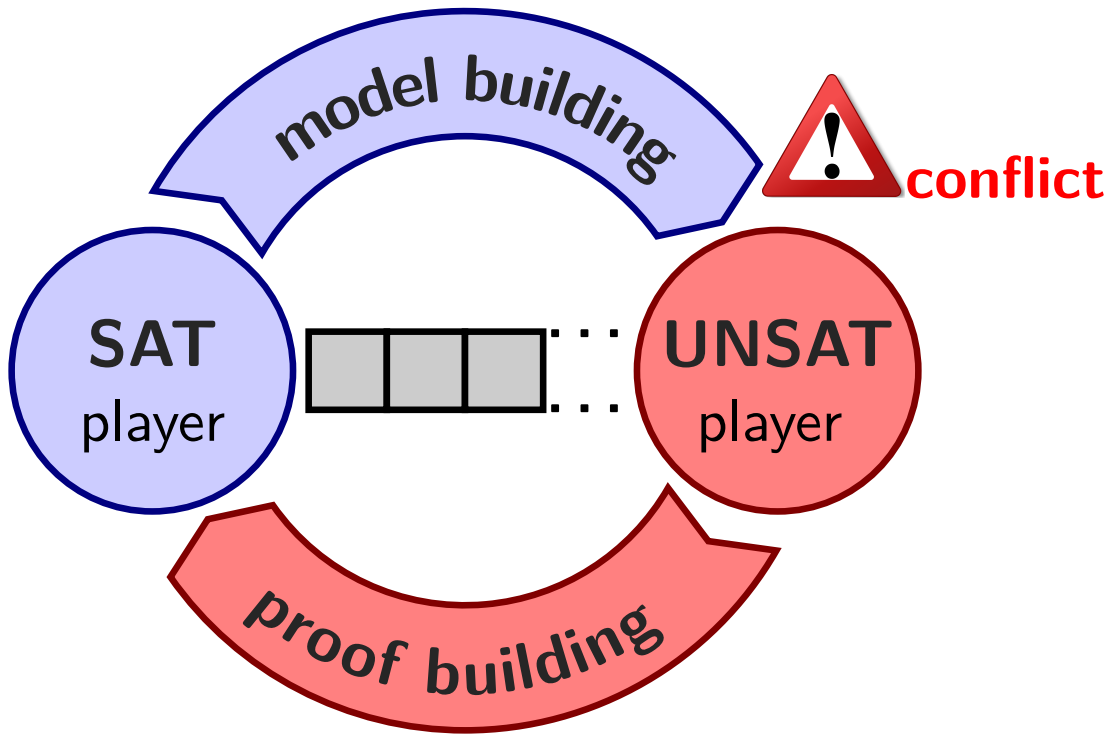

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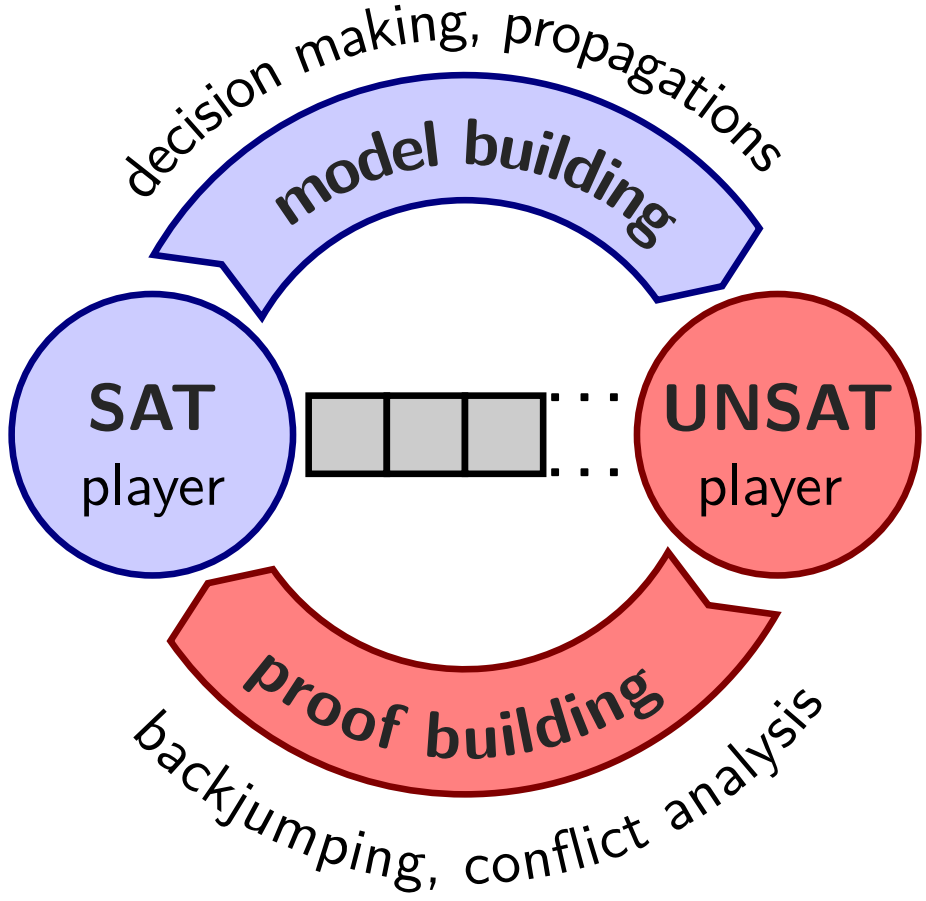

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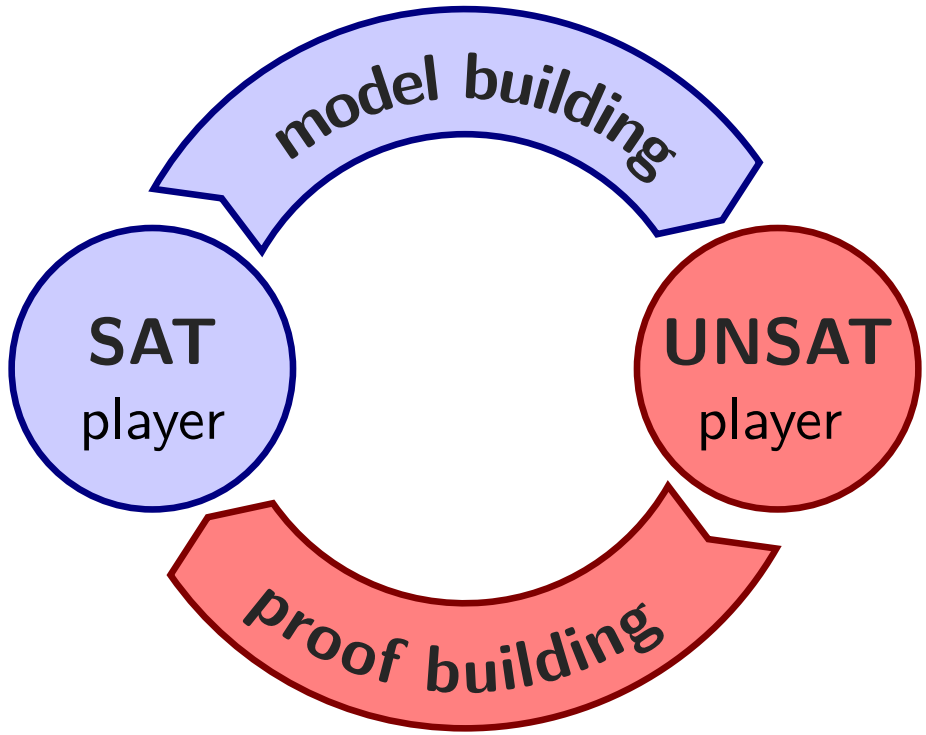

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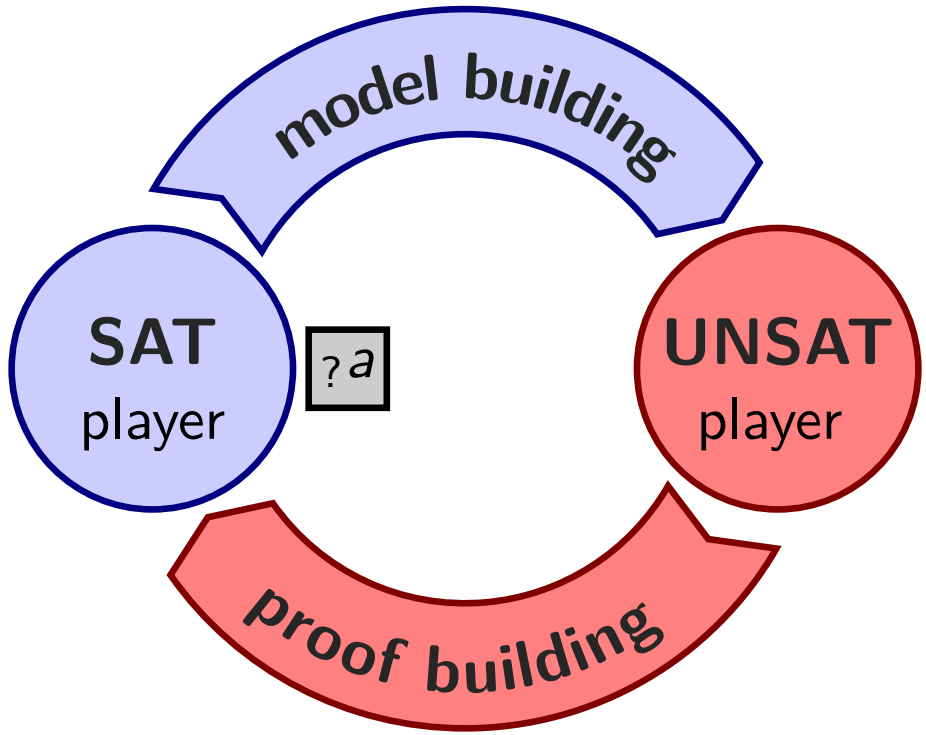

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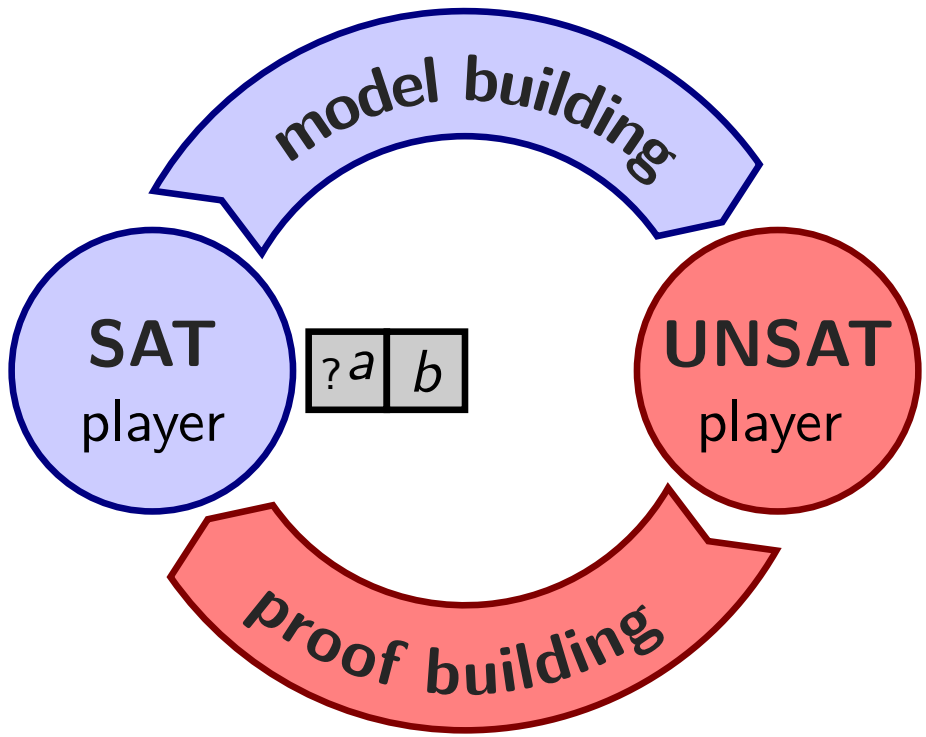

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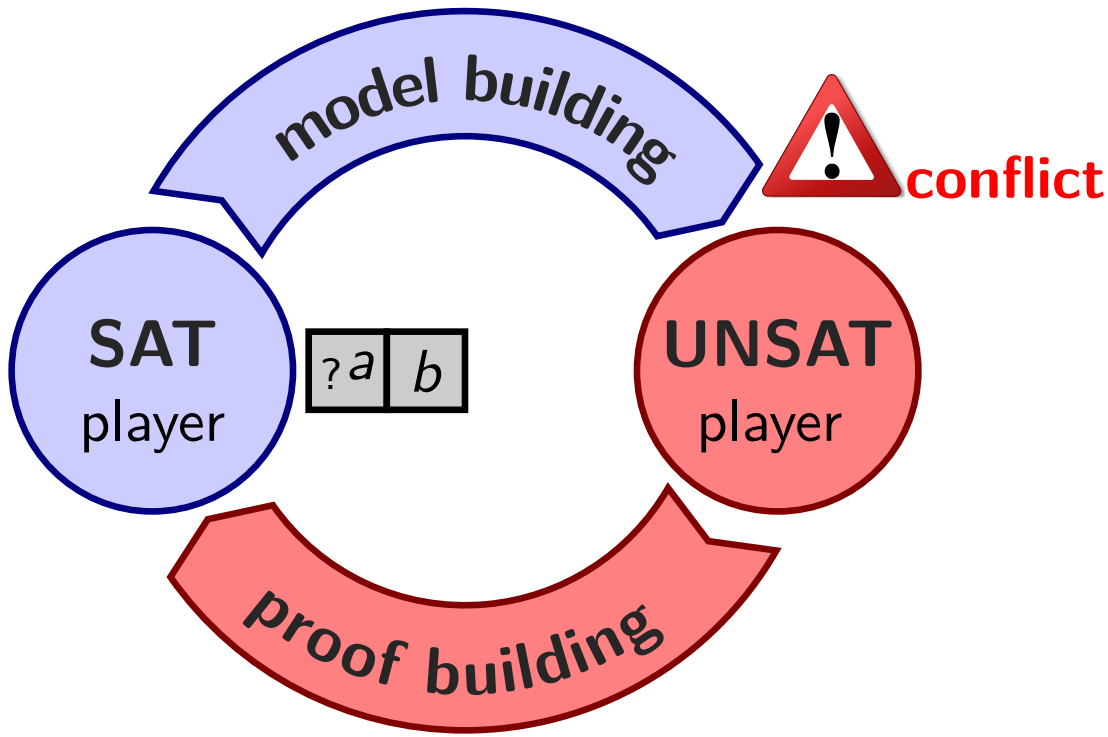

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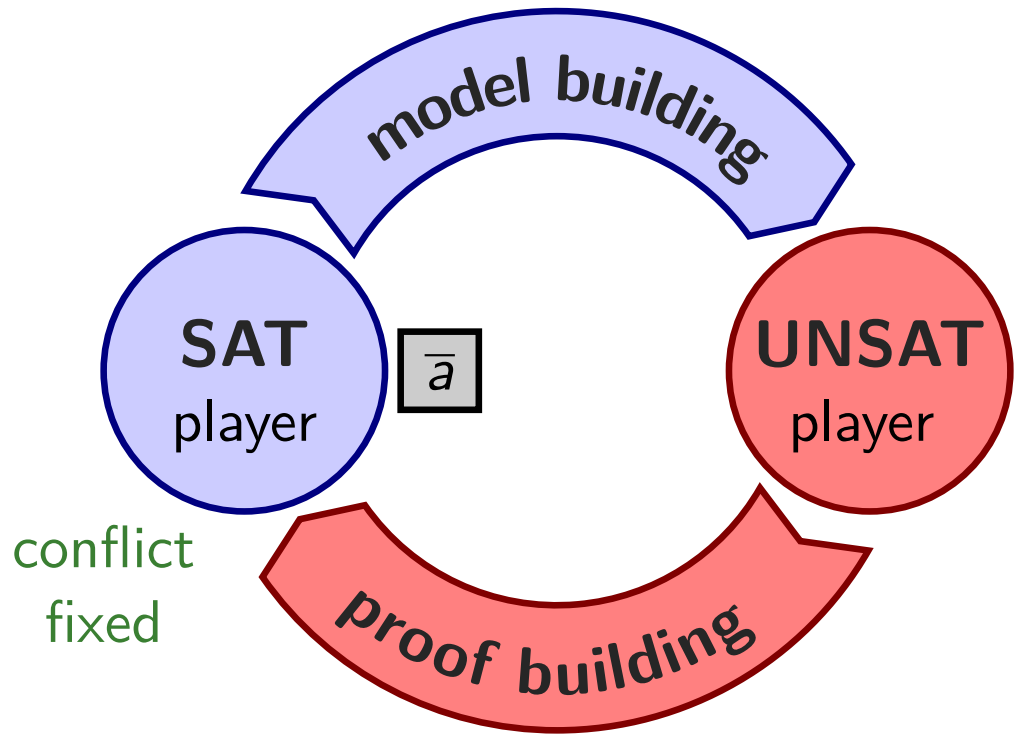

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2-player game to determine whether a formula is satisfiable.
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Archetype of conflict-driven reasoning: **CDCL**
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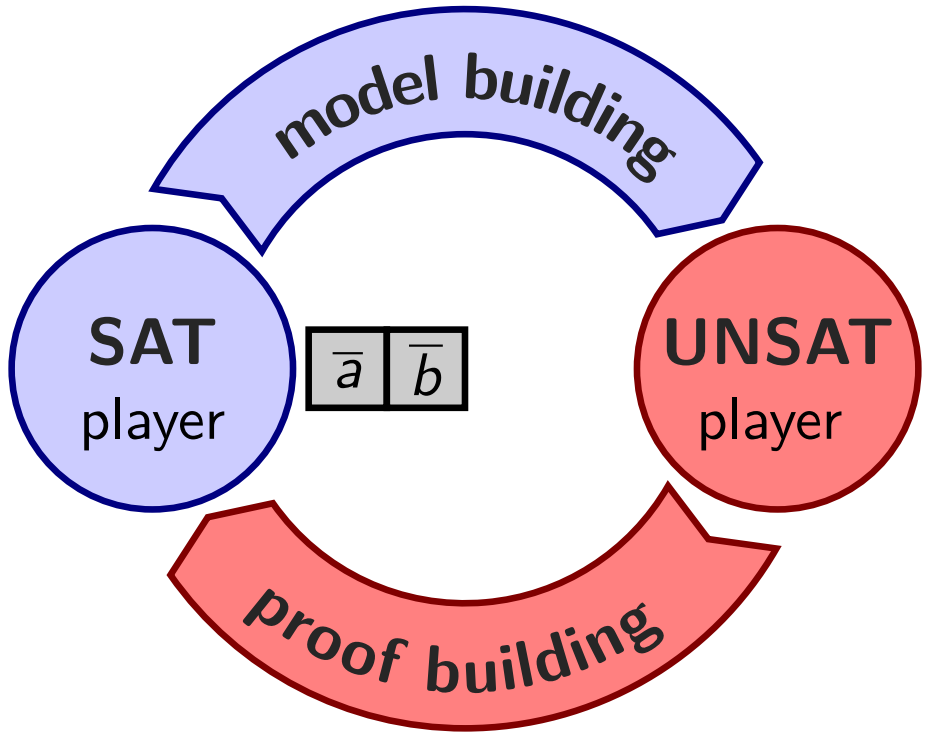

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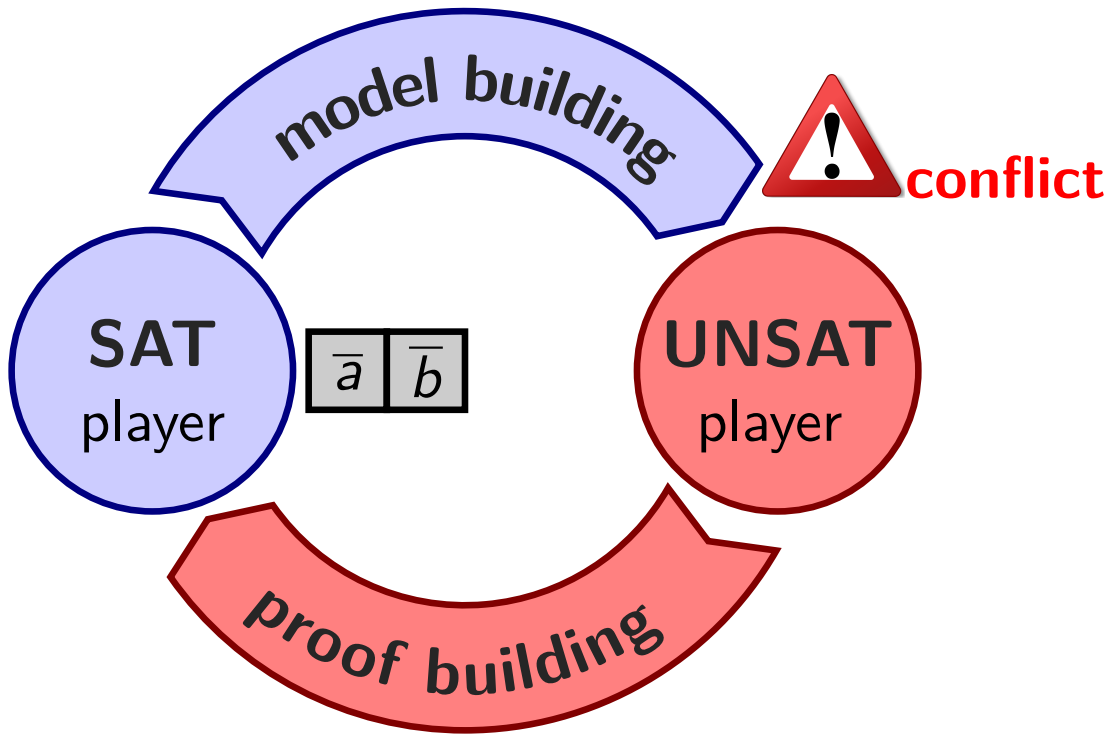

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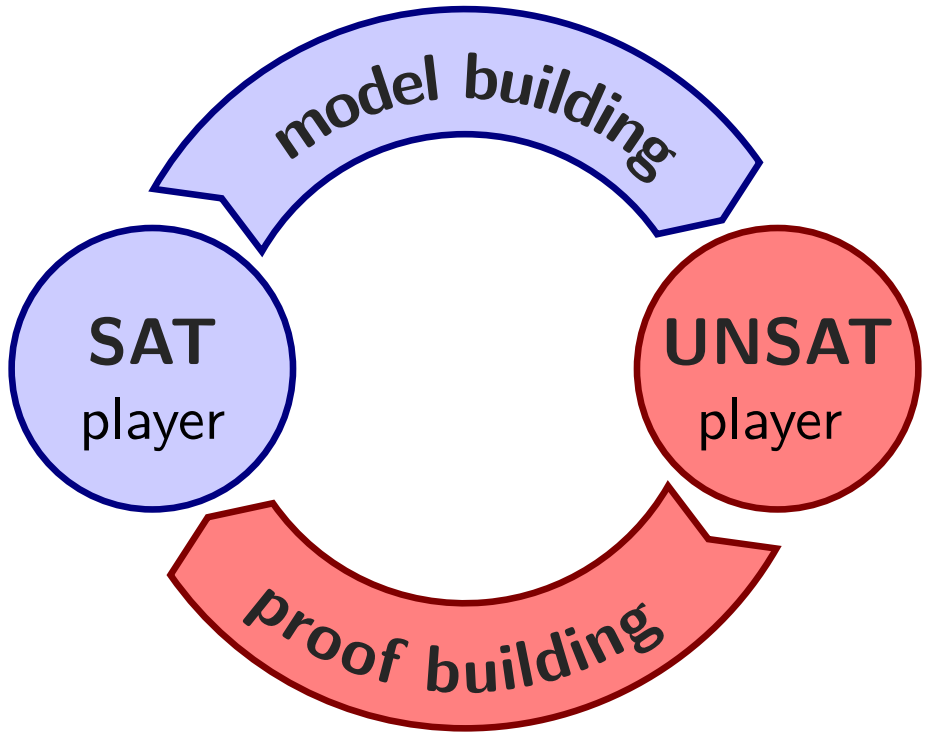

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$$\overbrace{(-2{\cdot}y - x < 0)}^{l_0}, \qquad \overbrace{(y + x < 0)}^{l_1}, \qquad \overbrace{(y < -1)}^{l_2}$$

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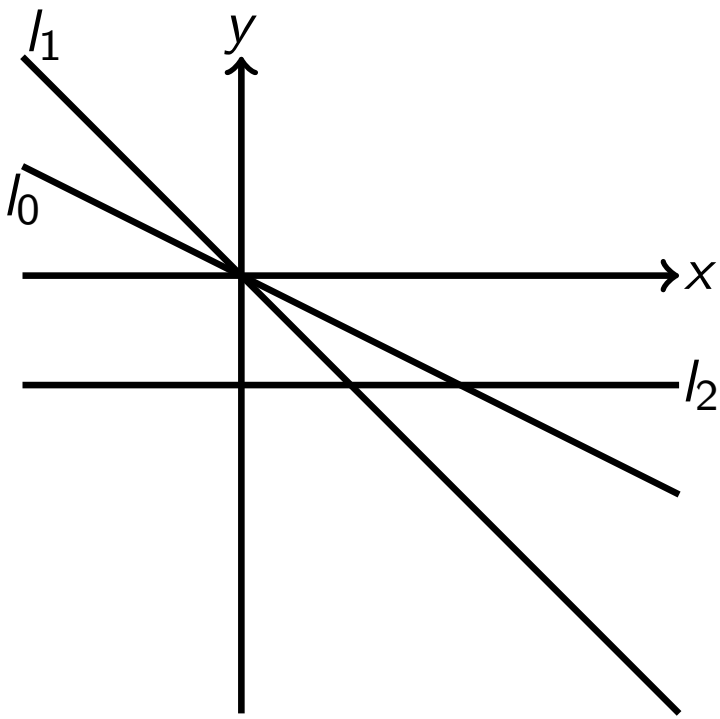

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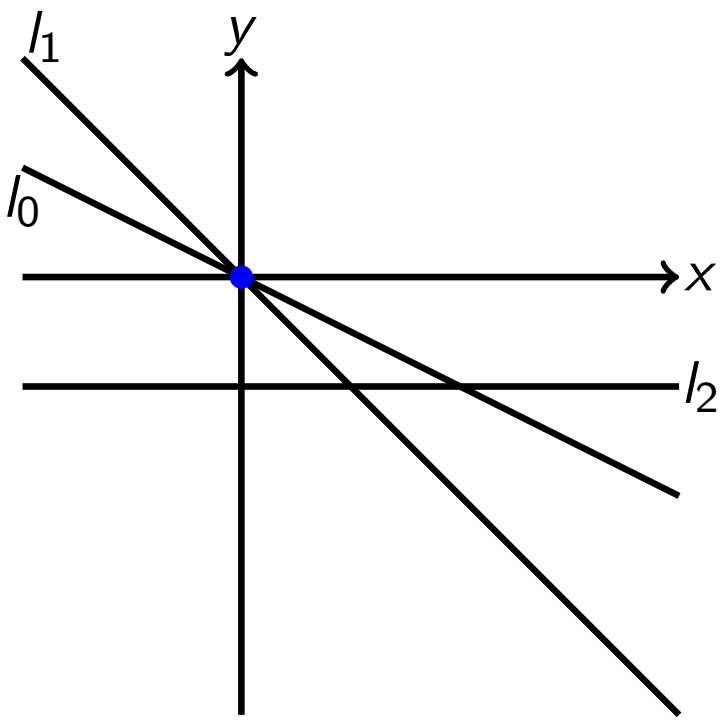

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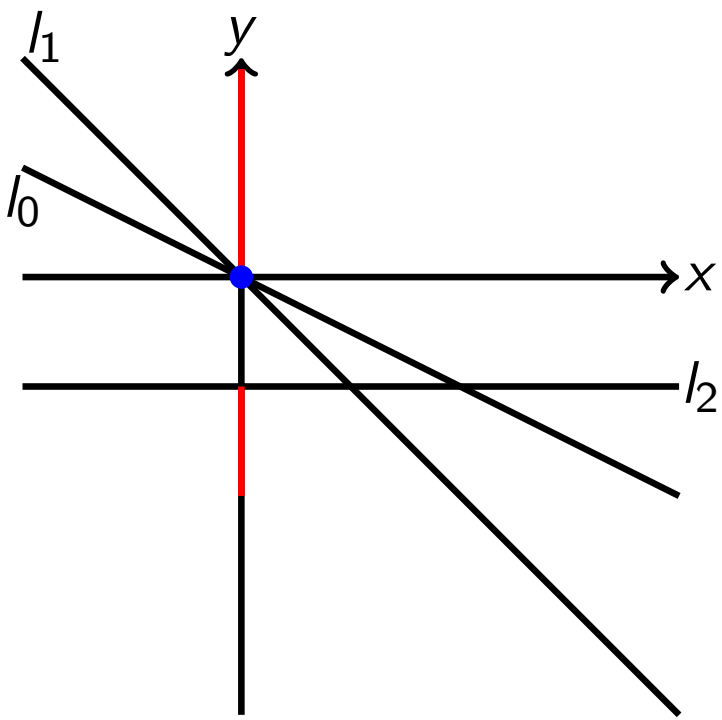

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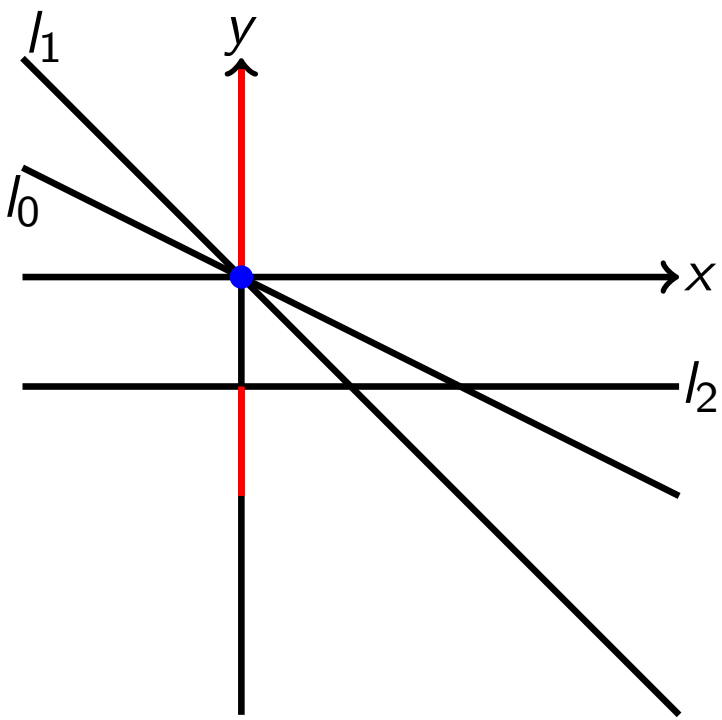


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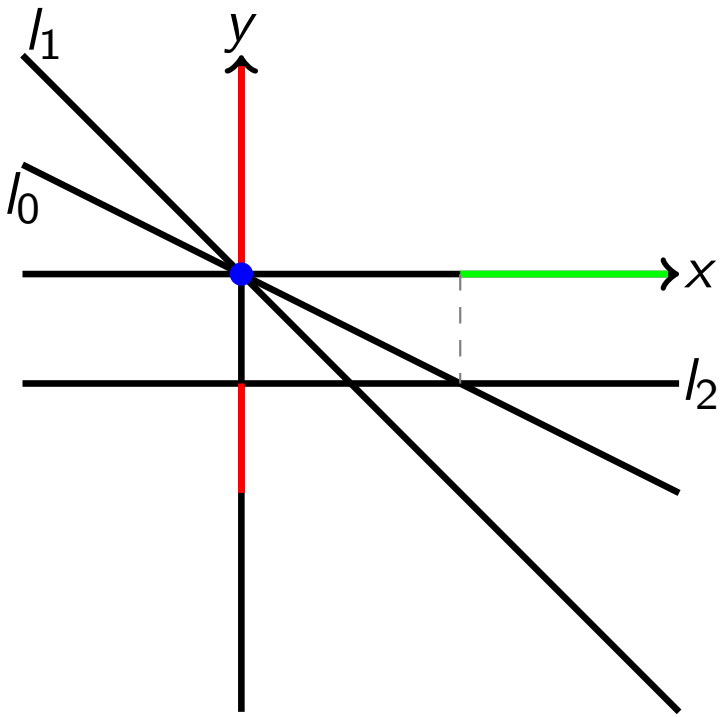


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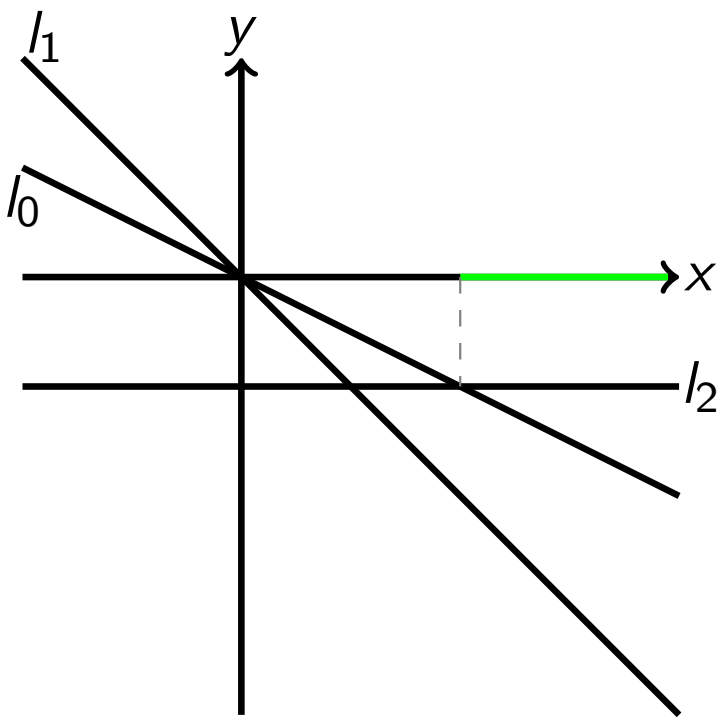


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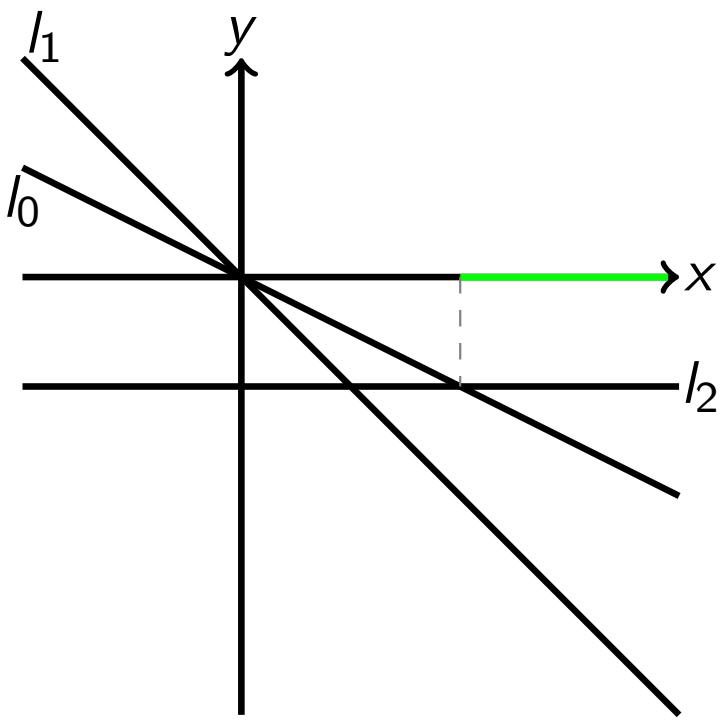


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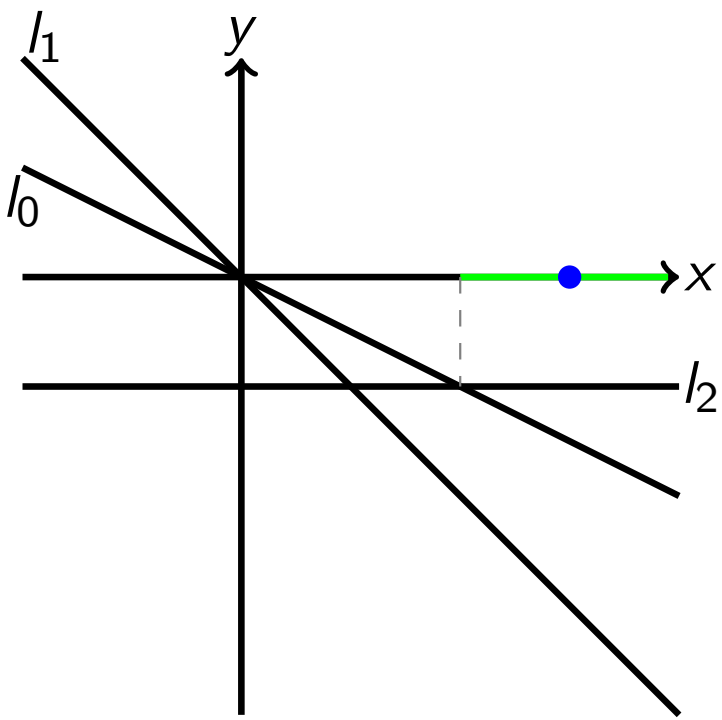

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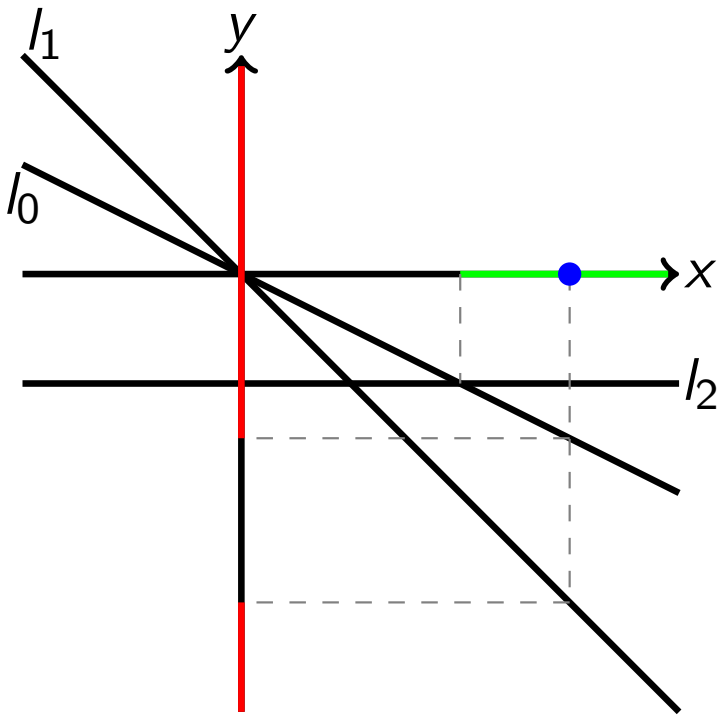


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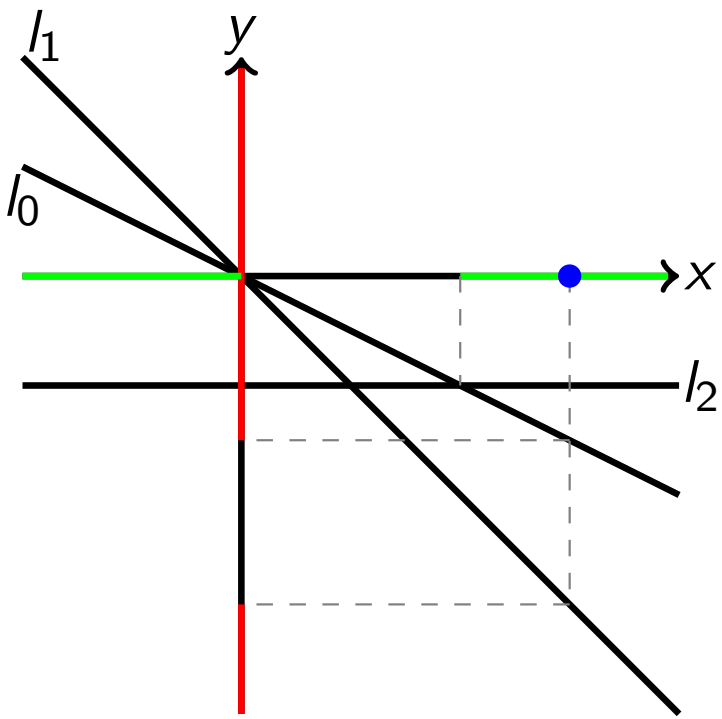


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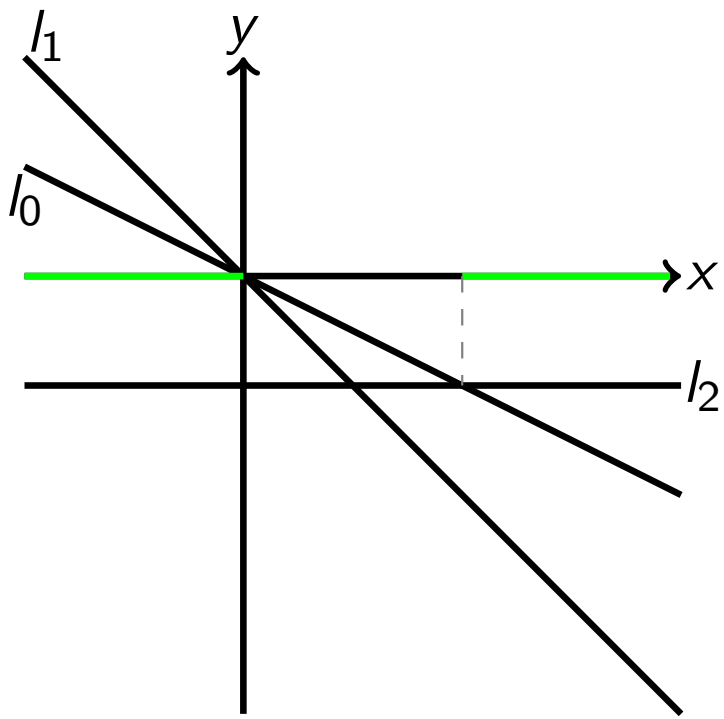


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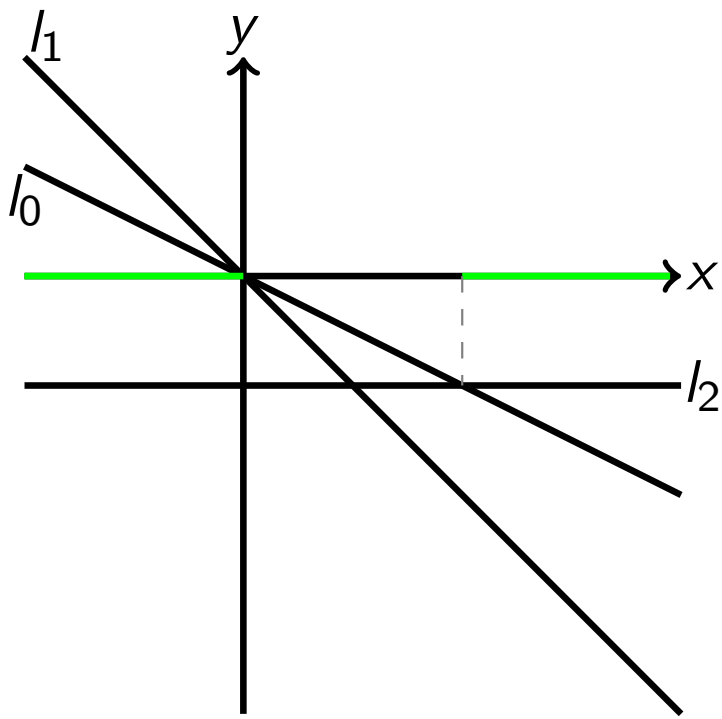


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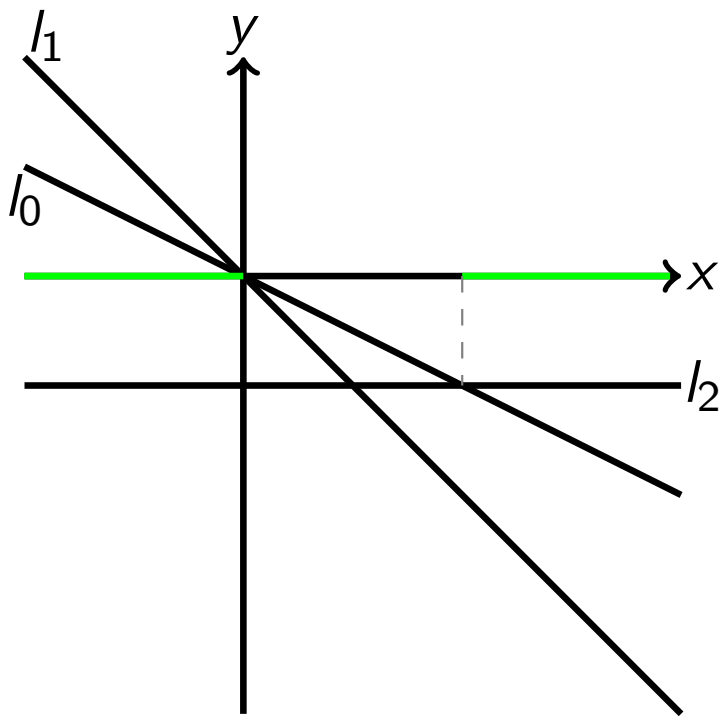


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# More generally

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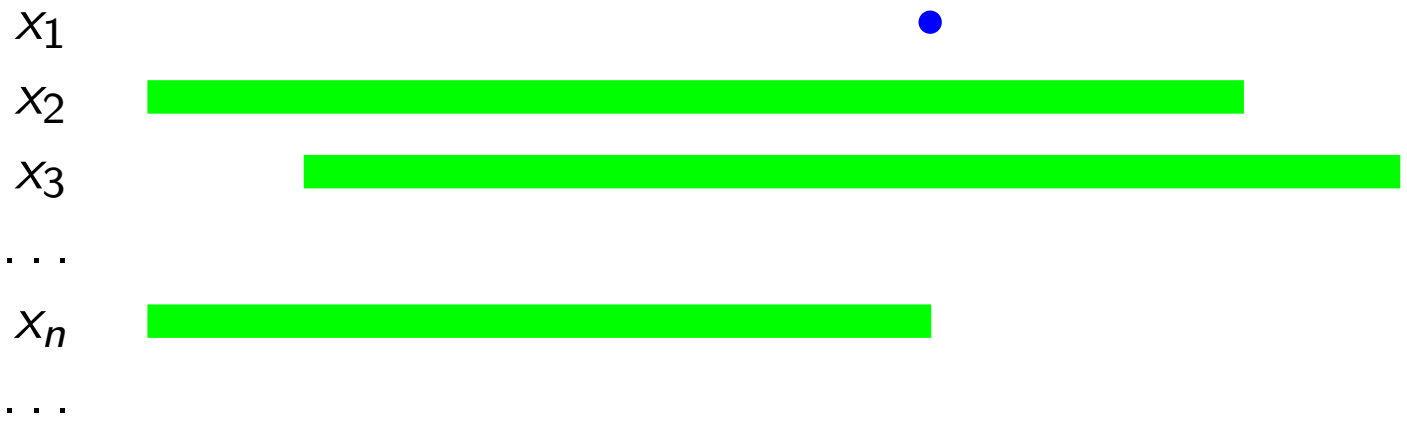


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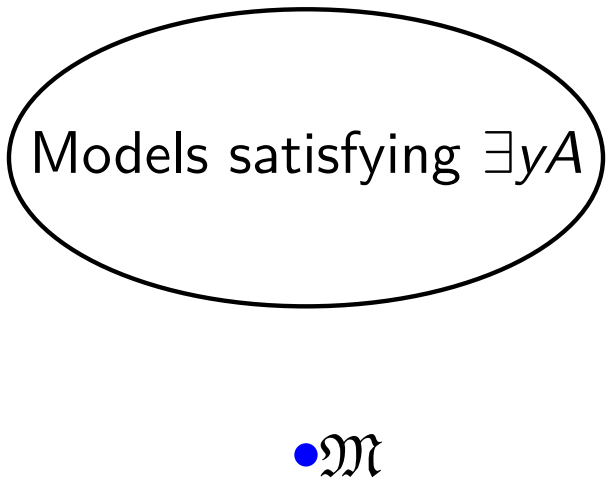

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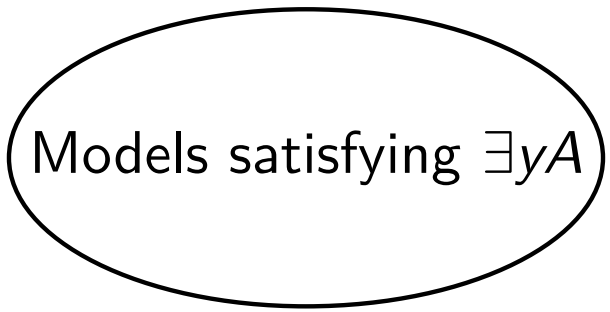


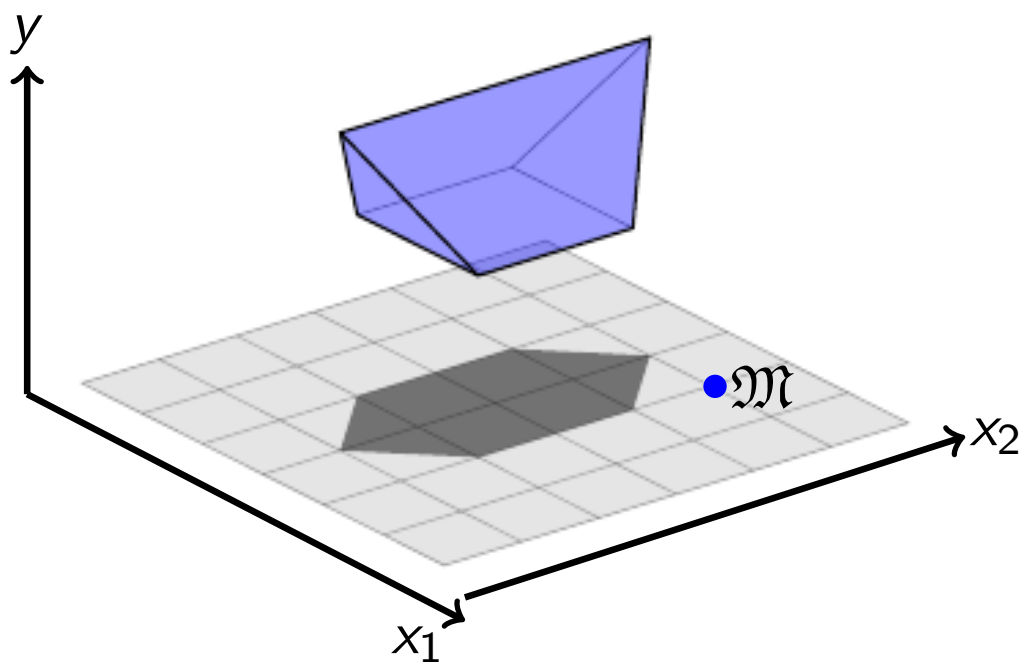

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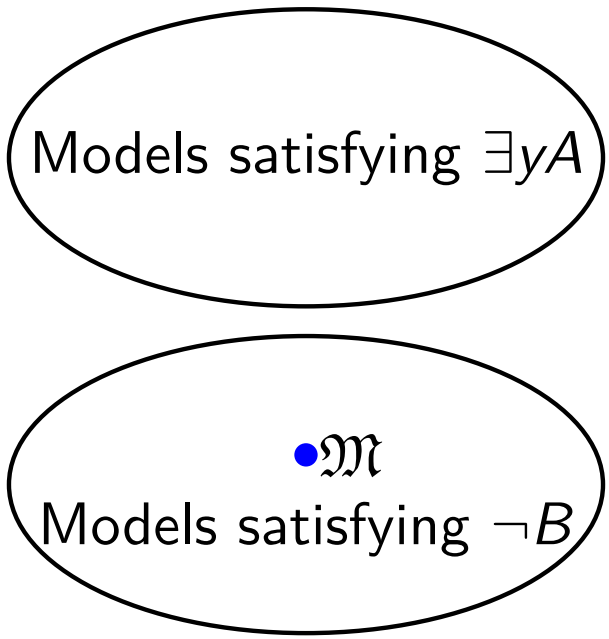


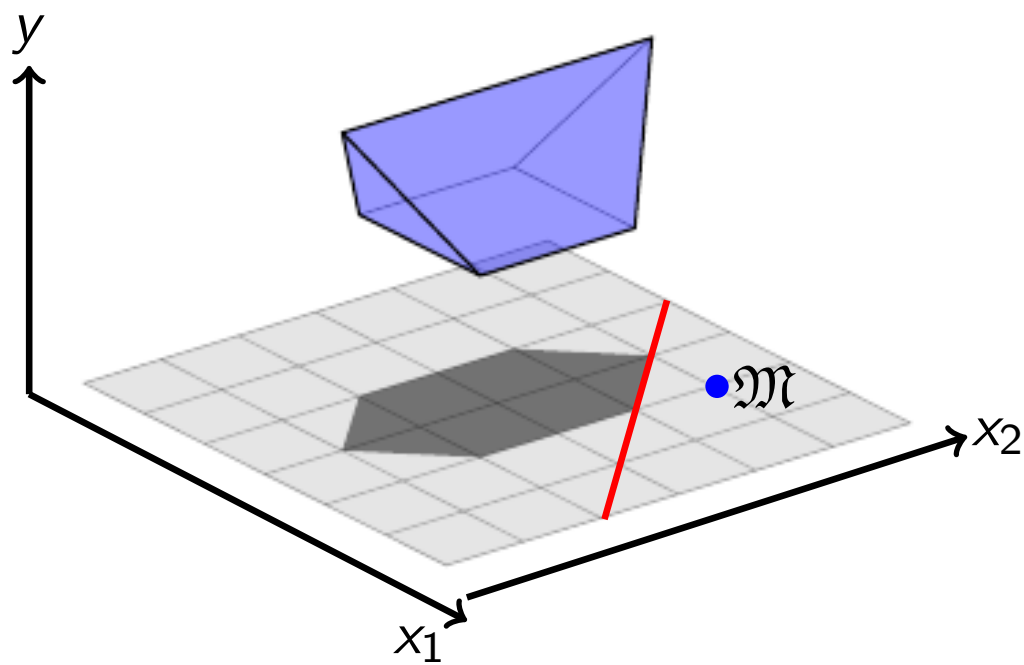


We seek to generalise $\mathfrak{M}$ into a class of models that do not satisfy $\exists y A$ "for the same reason" $\mathfrak{M}$ does not.

# The theory lemmas 2/2

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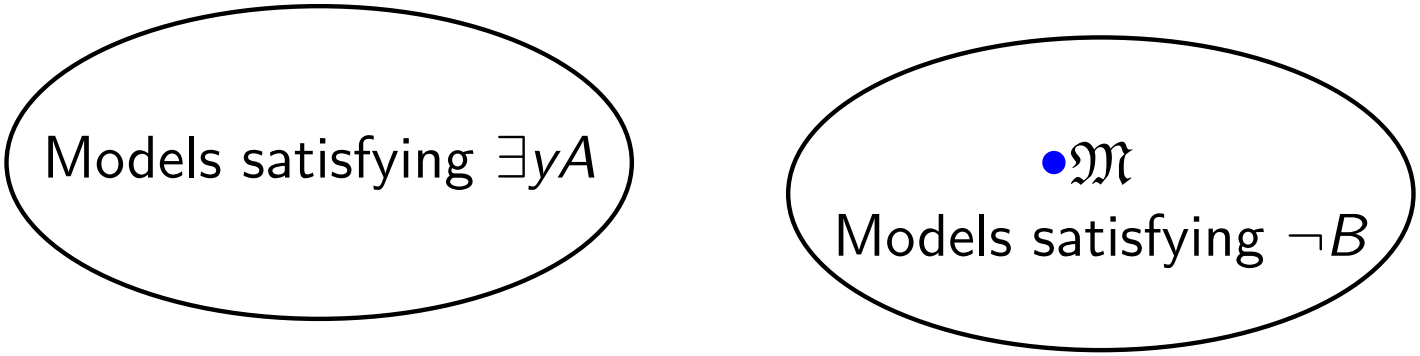

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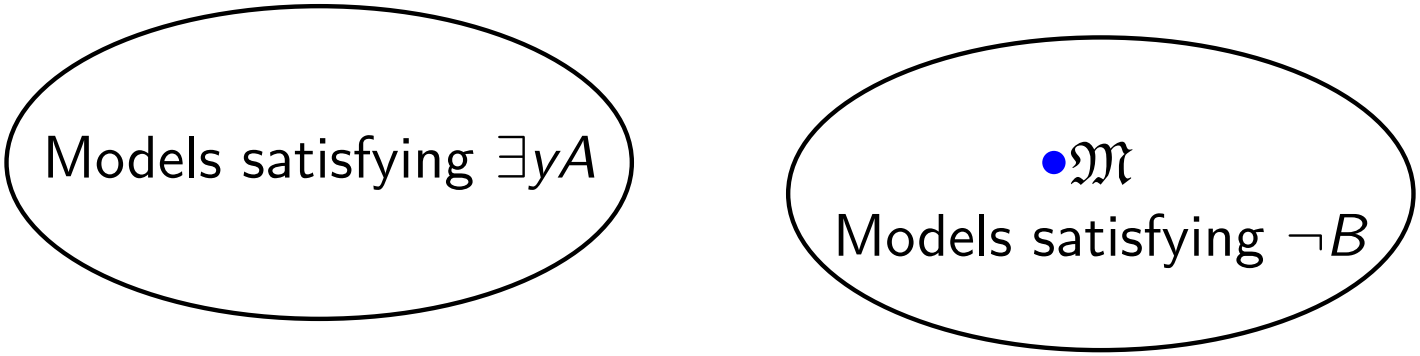


We characterise this class as those models not satisfying $B$, for some quantifier-free $B$ (with $\mathrm{fv}(B) \subseteq \{\overrightarrow{x}\}$) such that

- $\mathcal{T} \models (\exists y A) \Rightarrow B$
- $\mathfrak{M} \not\models B$

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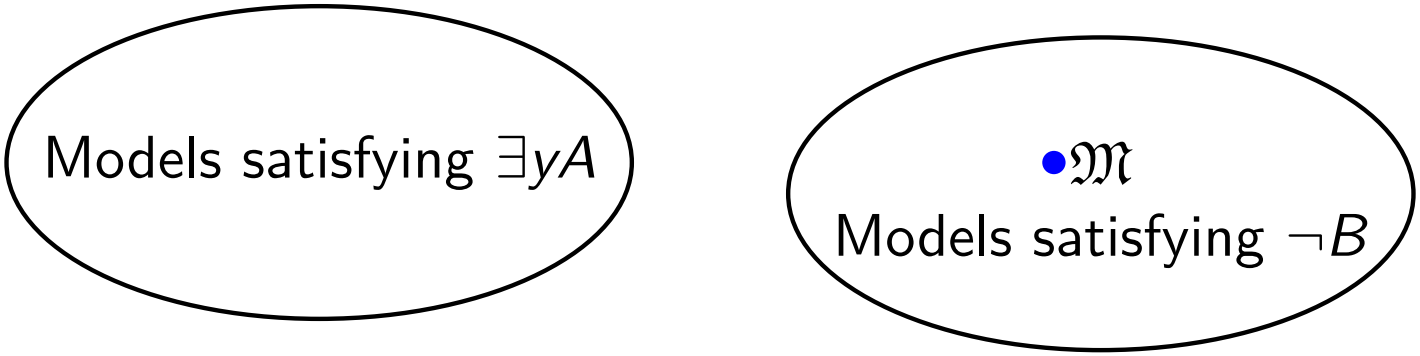


We characterise this class as those models not satisfying $B$, for some quantifier-free $B$ (with $\mathrm{fv}(B) \subseteq \{\overrightarrow{x}\}$) such that

- $\mathcal{T} \models (\exists y A) \Rightarrow B$
- $\mathfrak{M} \not\models B$

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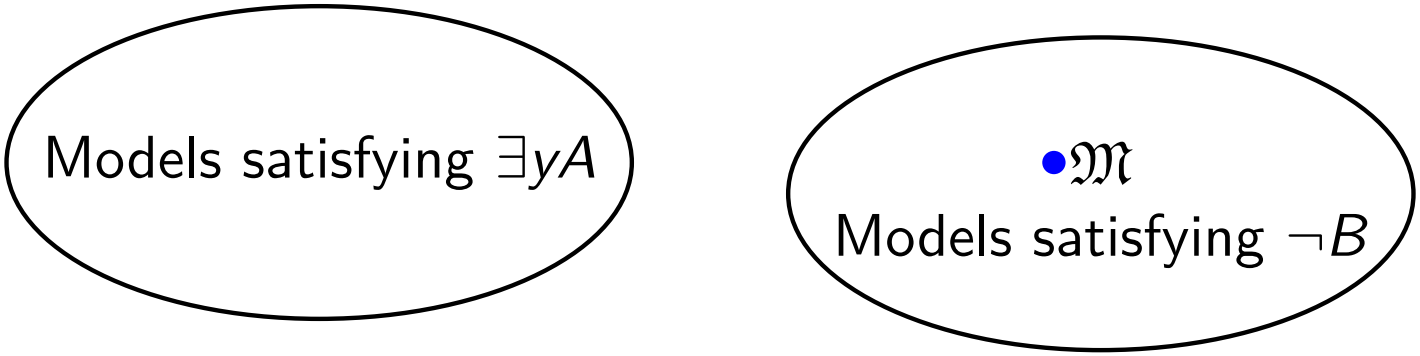


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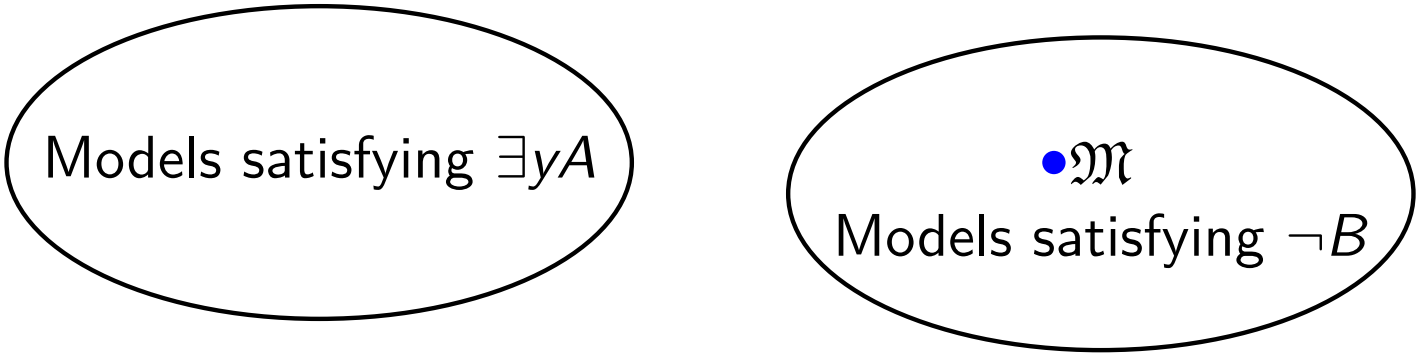


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For LRA, $B$ is provided by Fourier-Motzkin resolution from only two of the unit constraints

$$\frac{e_1 - y \lessdot_1 0 \quad e_2 + y \lessdot_2 0}{e_1 + e_2 \lessdot_3 0}$$

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In fact

- $B$ may not take all of the constraints in $A$ into account (just enough to defeat $\mathfrak{M}$)
- $A$ itself does not contain all constraints involving $y$ in the full problem, only those that $\mathfrak{M}$ has made unit in $y$

An objective is to make the production of $B$ as cheap as possible

# In practice

In non-linear arithmetic, computing the projections for quantifier elimination is done by Cylindrical Algebraic Decomposition (CAD).
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Rather expensive if full projections are computed
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Using partial / model-guided projections (i.e., lazy quantifier elimination) turns out to be a good approach in practice.
Implemented in the Yices SMT-solver by Jovanović
(probably the best solver for non-linear arithmetic)

# Bit vectors

Similar situation with the theory of bitvectors: $(y||x) <_u (y<<3)$

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- A management of ranges using Binary Decision Diagrams

  $x_1$

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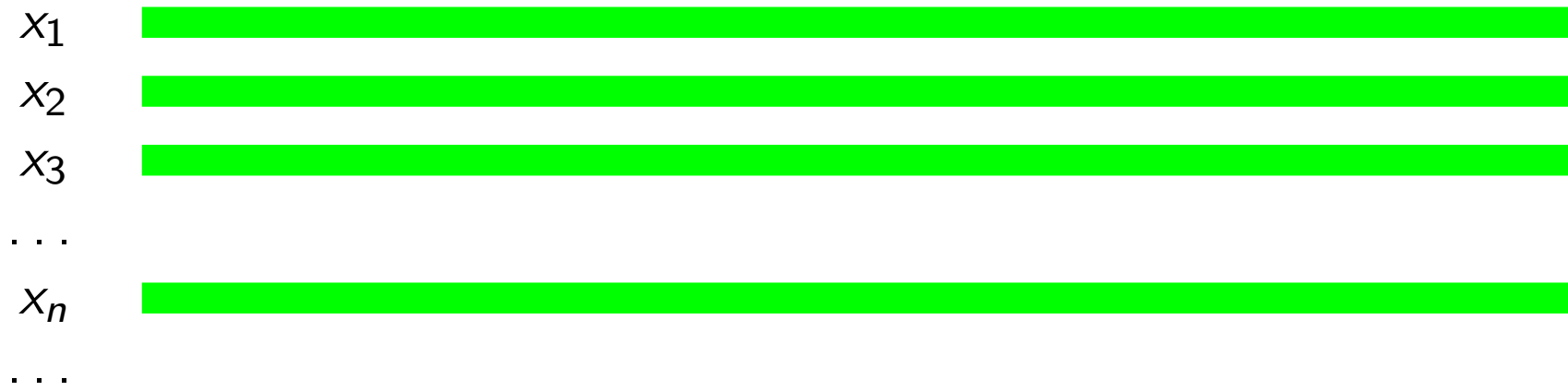


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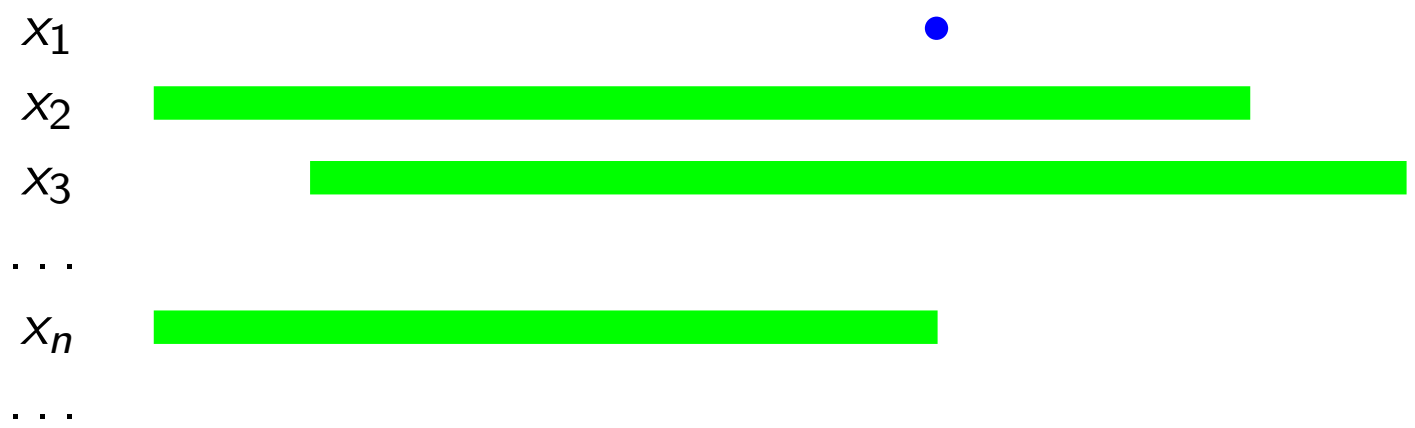


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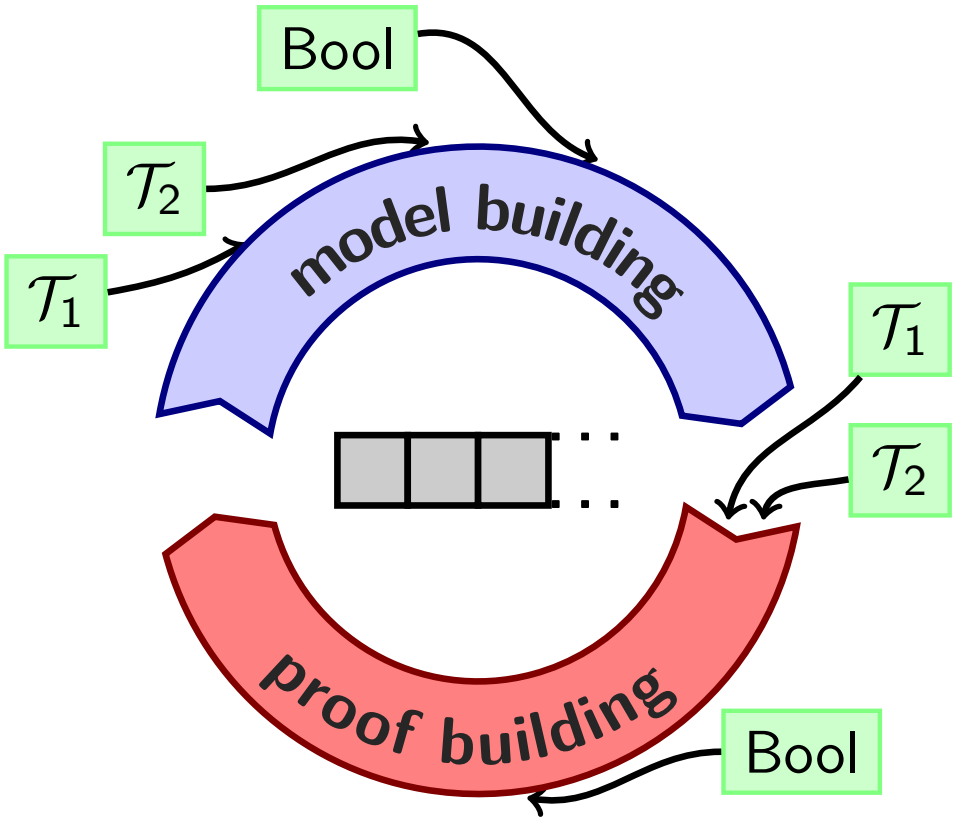
# In CDSAT

In CDSAT [BGLS17, BGLS17] we propose a comprehensive framework for conflict-driven reasoning modulo several (but disjoint) theories.
The theory combination is organised directly in the main conflict-driven loop:



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Can we have a lazier approach as in the existential case?

The basic quantifier elimination algorithm, which treats $\forall$ as $\neg\exists\neg$, heavily relies on the elimination of a quantified variable being an equivalence.
We can anticipate that the weaker condition $\mathcal{T} \models (\exists y A) \Rightarrow B$ required in MCSAT may be insufficient when treating alternations of $\forall$ and $\exists$

# The approach by Bjørner and Janota 1/3

In [BJ15], Bjørner and Janota propose an algorithm for "playing with quantified satisfaction", inspired by QBF.

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The SAT and UNSAT players from MCSAT exchange roles at every quantifier alternation:

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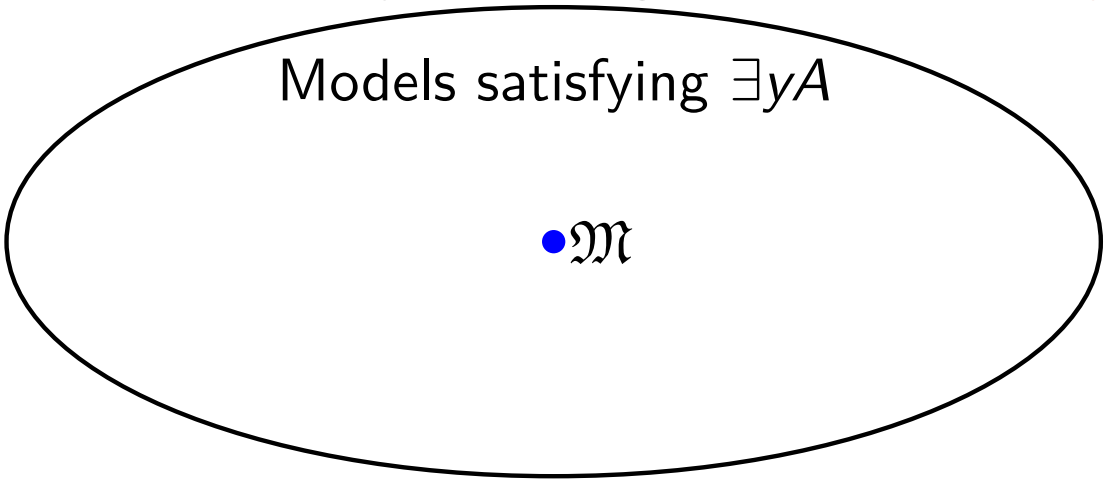
In order to determine whether $\forall x \exists y A$ is satisfiable:

- UP first chooses a value for $x$;
- If EP cannot find a value for $y$ to make $A$ true, UP has won.
- If on the contrary EP can find a value, UP needs to understand what was wrong about its choice for $x$.

Relies on theory $\mathcal{T}$ having model-based projection

# The approach by Bjørner and Janota 2/3

Relies on theory $\mathcal{T}$ having model-based projection



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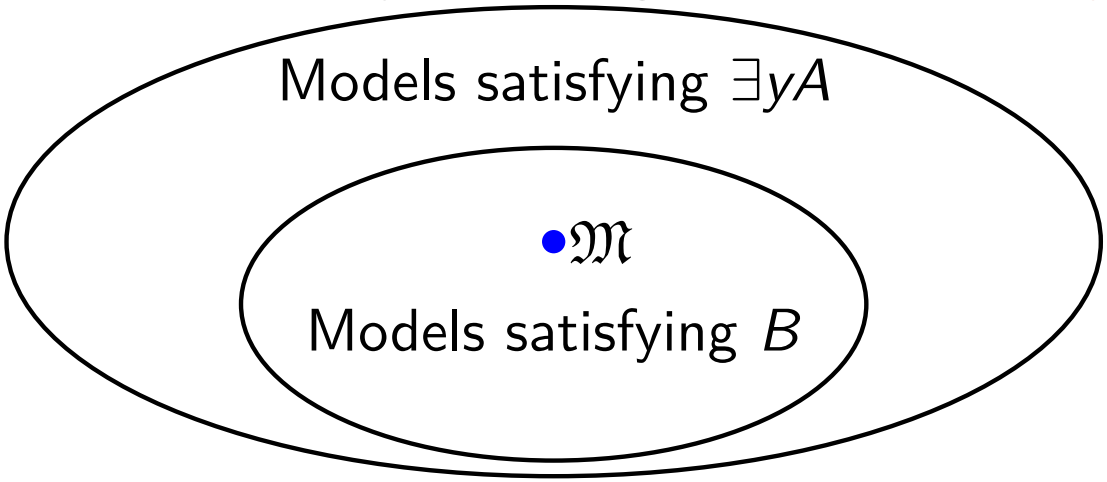


From $\mathfrak{M} \models \exists yA$,
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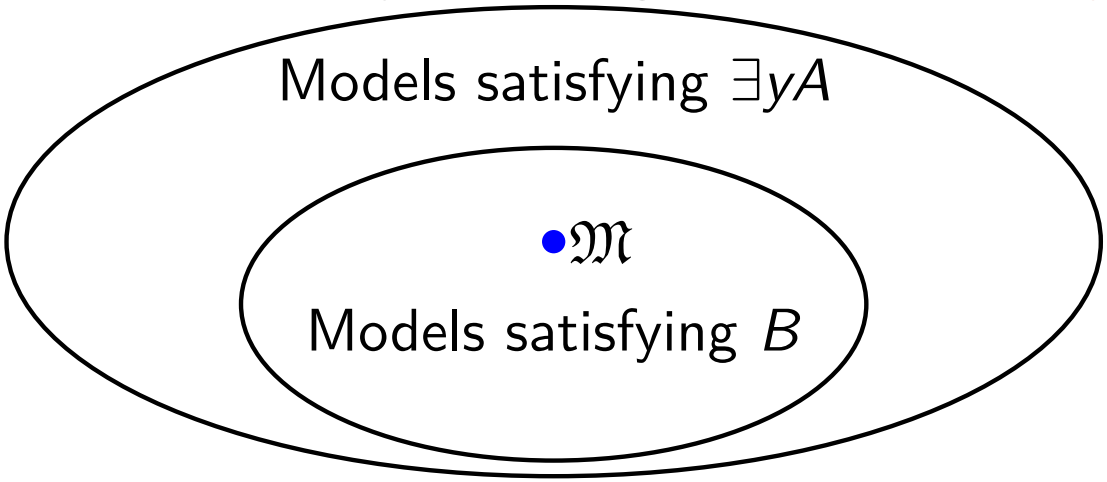


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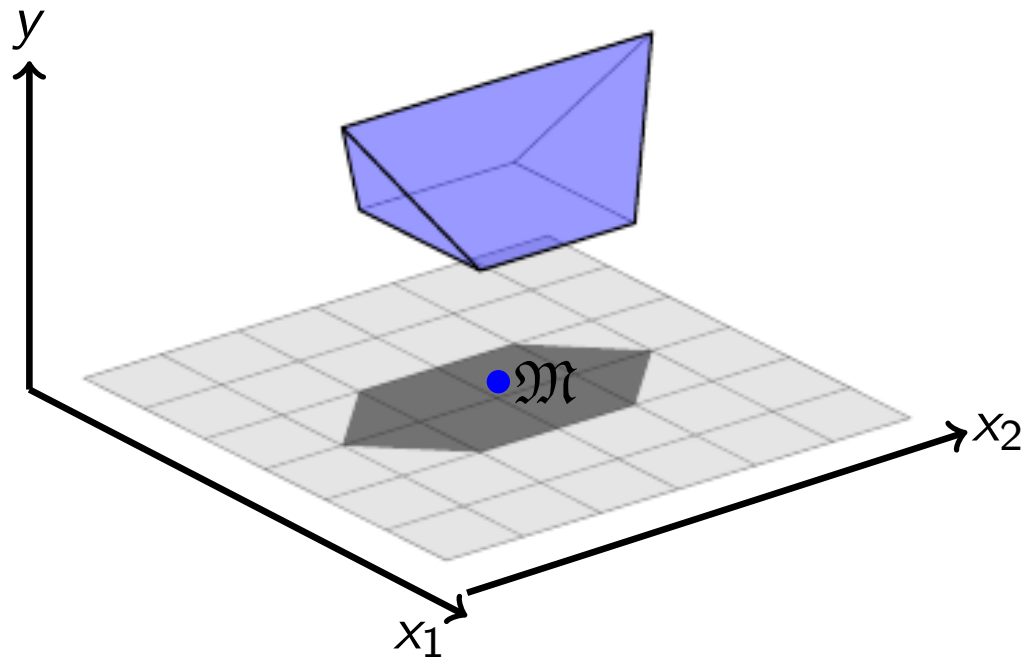

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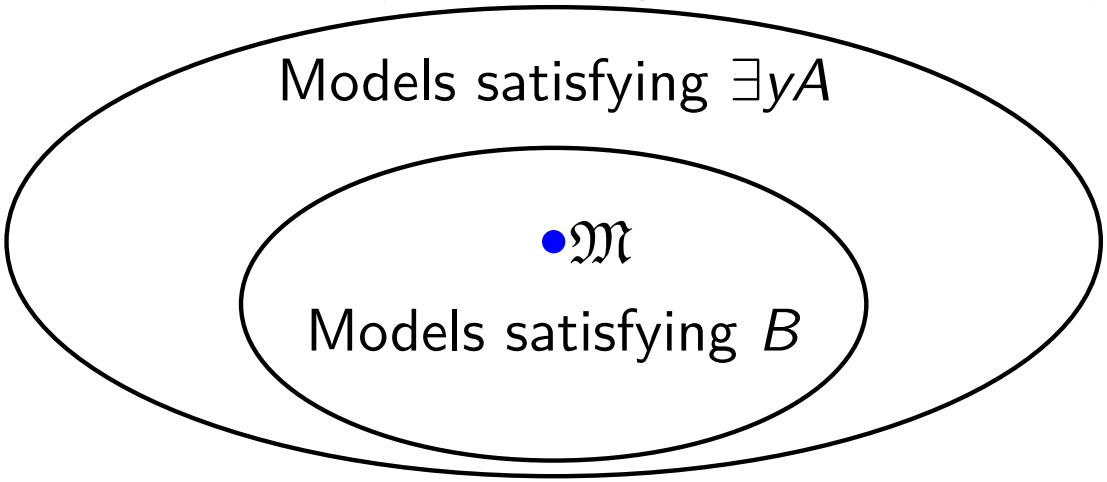


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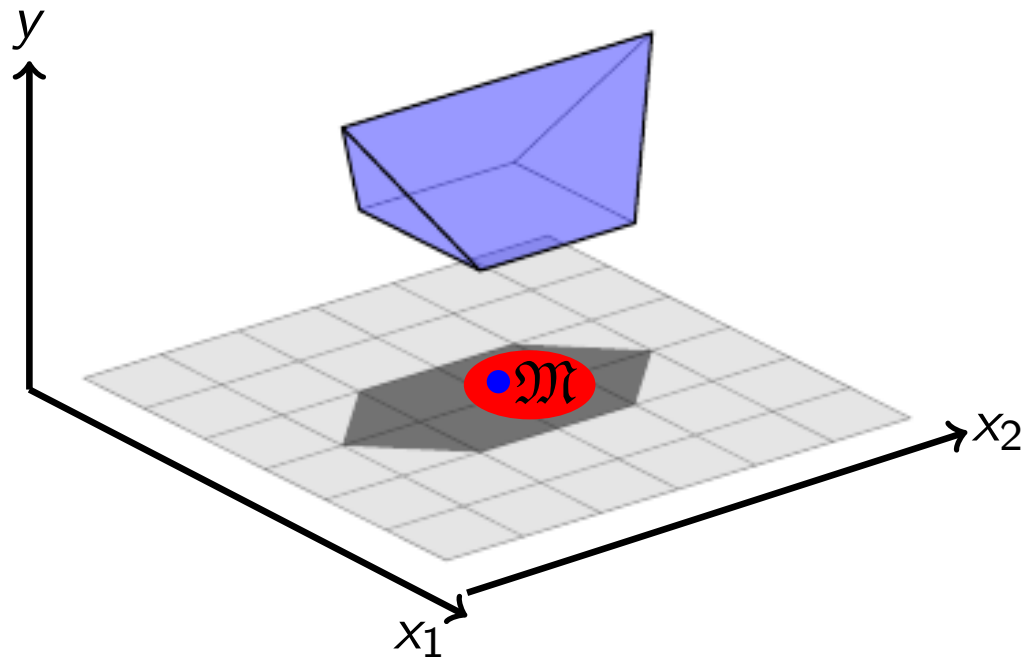

# The approach by Bjørner and Janota 3/3

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And $\exists y A$ should be covered by finitely many $B$'s:
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specific form of quantifier elimination

From there, an algorithm is given that applies the mbp computations lazily.

# Questions:

On purely existential problems, does the Bjørner-Janota algorithm collapse to MCSAT?
I suspect not: the form of laziness seems distinct, but it also seems complementary.

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Ongoing work:

- extend MCSAT with the Bjørner-Janota approach for handling arbitrary alternations of quantifiers.
- what is the notion of proof production in this approach?

# Related work and future work

Investigate related approaches:

- The ANR Decert work on Linear Integer arithmetic, which extends Fourier-Motzkin with simplex-based techniques [BCC+12]
- Monniaux's work on quantifier elimination [Mon08, Mon10]. It uses a ground SMT-solver as a black box (for purely existential problems), and also performs some QE-elimination steps (e.g., FM resolutions) independently from the SMT-solver.
- Dutertre's work on solving "EF problems" ($\exists\forall$) in Yices, also relying on a ground SMT-solver considered as a black box.

How would the Bjørner-Janota approach work in a combination of theories?
Just as our CDSAT system generalises MCSAT to a combination of theories, what would be the equivalent for the Bjørner-Janota approach?

Questions?


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