About Nonlinear Sampled-data Dynamics Representations

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Motivations

Continuous-time Dynamics:

$$\dot{x}(t) = f(x(t), u(t)) \quad y(t) = h(x(t)); \quad f : R^n \times R^m \to R^n; h : R^n \to R^p$$

Discrete -Time Dynamics

$$x_{k+1} = F(x_k, u_k); \quad y_k = H(x_k); \quad F : R^n \times R^m \to R^n; H : R^n \to R^p$$

Sampled-data Dynamics

$$x_{k+1} = F^\delta(x_k, u_k); \quad y_k = H^\delta(x_k) \quad F^\delta, H^\delta \text{ parameterized by } \delta \in]0, T^*]$$

$u(t) = u_k = Cst; t \in [k\delta, (k+1)\delta[$ ZOH; $k = (0, 1, ...); \delta$:*sampling period*

Find equivalent representations : A unified mathematical framework

Hybrid Dynamics as

- *CT and DT Dynamics with Switches or Jumps*

- *Interconnection of CT and DT Dynamics - Networks*

- *Sampled-data Dynamics in a Computer Aided Environment*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_k) \qquad & u(t) = u_k = Cst; t \in [k\delta, (k+1)\delta[\\ x^+ &= x_k \qquad & t = k\delta; k = 0, 1, ... \end{aligned}$$

- *Variable Sampling : Event-triggered dynamics*

vast literature, variety of concepts, approaches, interdisciplnary

Sampling : a Unifying Tool

- *To get a unified representation and unified mathematical framework*
- *To deal with Computer Aided analysis and design - unavoidable*
- *To make feasible the CT solutions - further to the Zero Order Holding device*
- *To provide new solutions taking advantage of sampling - multirate approaches*
- *To provide computable solutions through constructive algorithms*

Outline of the presentation

- *Nonlinear sampling and equivalent sampled-data representations*
- *Exponential form representations of the behaviors*
- *A new Differential-Difference representation of DT Dynamics - DDR*
- *Some Impacts in Control Theory*
- *Some Computational Open Questions*

— *From chronological calculus to exponential representations of continuous and discrete-time dynamics: a Lie algebraic approach, TAC, 52, 2227-2241, (2007). S. Monaco, D. Normand-Cyrot and C. Califano*

— *Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control, Special Issue "Fundamental Issues in Control" , EJC, 13 (2,3), 221-241, (2007). S. Monaco, D. Normand-Cyrot*

Nonlinear sampling and equivalent sampled-data representations

Given: $\dot{x}(t) = f(x(t), u(t))$ *CD*

with: $u(t) = u_k = Cst$ *ZOH*

(over $[k\delta, (k+1)\delta[$; $k = (0, 1, ...)$; δ : sampling period)

Find a Nonlinear Difference Equation:

$$x_{k+1} = F^{\delta}(x_k, u_k)$$

such that: $x_k = x(t = k\delta)$ *when* $x_0 = x(0)$

(input-state reproduction at the sampling instants)

(sampled- data equivalent model)

• *Extensions: Multi-Input, Higher Order Holding device, Parameter or Time Varying Dynamics, ...*

- *formulate sampling as formal ODE Integration:* $\dot{x}(t) = f(x(t), u(t))$

- *derive the sampled-data equivalent model*

$\rightarrow$ in a computable state-space form through formal power series expansions in $\delta \rightarrow$ easy approximations

$\rightarrow$ in terms of the VF (f, g) and their Lie brackets $(ad_f^i g, ad_g ad_f^i g, ...)$: basic tools for characterizing structural and control properties in continuous-time

- *in the form of two coupled differential difference equations rather than in the form of a map*

$\rightarrow$ more suitable to characterize structural and control properties.

The mathematical framework

→ *Formal power series*
→ *Exponential series and Lie algebraic methods*
→ *Chronological calculus and combinatorics ...*

→ in Mathematics : *Ree (1958), Magnus (1954), Chen (1961), Grobner (1973)*, ...

→ in the Control theory (from about 40 years in CT - more recently in DT): *Agrachev, Gamkrelidze (1978), Sussmann (1984), Fliess (1986), Tretyak (1998), Kawski (1999), Sarychev (2001), ...*

→ in Computer Sciences

some notations

- Vector fields (complete) and functions smooth in x and analytic t and/or u
- the Lie derivative of λ along θ at x:

$$L_\theta \lambda|_x := \frac{\partial \lambda}{\partial x}|_x \theta(x) = \theta(\lambda)(x)$$

- $L^p_\theta \lambda := L_{\theta \circ \ldots \circ} L_\theta \lambda$; $p-$ *times* $L^0_\theta = 1$ *the identity operator*
- the exponential Lie series wrt L_θ :

$$e^{L_\theta} = 1 + \sum_{p \geq 1} \frac{1}{p!} L^p_\theta = e^\theta; \quad e^\theta h|_x = h|_{x=e^\theta x}$$

- the Lie bracket: $[\theta, \sigma] := \mathrm{ad}_\theta \sigma = L_{\theta \circ} L_\sigma - L_{\sigma \circ} L_\theta$
- for $p > 1$: $\mathrm{ad}^p_\theta(\sigma) := [\theta, \mathrm{ad}^{p-1}_\theta(\sigma)]$
- all the results are formal, no specific convergence study

single Input-Affine dynamics for simplicity:

$$\dot{x}(t) = f(x(t)) + u_k g(x(t))$$

Uncontrolled Dynamics - $u = 0$

$$\begin{aligned} \dot{x}(t) &= f(x(t)); \quad x(t = k\delta) = x_k \\ x_{k+1} &= x_k + \int_0^\delta f(x(\tau))\mathrm{d}\tau = e^{\delta f} x_k \\ e^{\delta f} x_k &= F^\delta(x_k) = x_k + \sum_{i \geq 1} \frac{\delta^i}{i!} F_i(x_k) \end{aligned}$$

The Linear Case : $\dot{x}(t) = Ax(t) \quad \to \quad x_{k+1} = A^\delta x_k = e^{\delta A} x_k$

Idem for Controlled Dynamics under ZOH

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_k) \qquad u(t) = u_k, t \in [k\delta, (k+1)\delta[\\ x_{k+1} &= x_k + \int_{k\delta}^{(k+1)\delta} f(x(\tau), u_k) \mathrm{d}\tau \\ e^{\delta f(.,u_k)} x_k &= F^\delta(x_k, u_k) = F_0^\delta(x_k) + \sum_{i \geq 1} \frac{u_k^i}{i!} F_i^\delta(x_k) \end{aligned}$$

- $F^\delta(., u_k)$: *exact sampled equivalent to* $f(., u_k)$ *(asymptotic expansion in δ)*

- *closed form solutions* do not exist in general

Sampled equivalent to I-A dynamics as a map

$$\dot{x}(t) = f(x(t)) + u_k g(x(t))$$

$$x_{k+1} = F^\delta(x_k, u_k) = e^{\delta(f+u_k g)} x_k$$

$\rightarrow$ *nonlinearity in* u : $F^\delta(., u) = F_0^\delta + \sum_{p\geq 1} \frac{u^p}{p!} F_p^\delta$

$\rightarrow$ *integro-differential formula of Poincaré* - *(MNC,1984)*

$$F_0^\delta(x) = e^{\delta f} x$$

$$F_p^\delta(x) = p \int_0^\delta e^{(\delta - s) f} L_g F_{p-1}^s(x) ds \qquad p \geq 1$$

The Linear Case

$$\dot{x}(t) = Ax(t) + Bu_k \qquad \rightarrow \qquad x_{k+1} = A^\delta x_k + B^\delta u_k$$

$$A^\delta = e^{\delta A} \;=\; I + \delta A + \frac{\delta^2}{2!}A^2 + \ldots + \frac{\delta^i}{i!}A^i + \ldots$$

$$B^\delta = \int_0^\delta e^{\tau A} B d\tau \;=\; \delta B + \frac{\delta}{2!}AB + \ldots + \frac{\delta^i}{i!}A^{i-1}B + \ldots$$

linearity in x and in u preserved

exponential of matrices : linear algebra

computational facilities

time-varying dynamics

$$\dot{x}(t) = f(t, x(t)) \quad = \quad f_0(x) + \sum_{i \geq 1} \frac{t^i}{i!} f_i(x)$$

integration $\downarrow$ *over* $[0, \delta[$

$$x(\delta) = \int_0^\delta f(\tau, x(\tau)) \mathrm{d}\tau \quad = \quad \overrightarrow{\exp} \int_0^\delta f(\tau, .) \mathrm{d}\tau |_{x_0}$$

the chronological exponential series solution :

$$\overrightarrow{\exp} \int_0^\delta f(\tau, .) \mathrm{d}\tau \quad = \quad 1 + \int_0^\delta f(\tau_1, .) \mathrm{d}\tau_1 + \sum_{m \geq 1} \int_0^\delta \mathrm{d}\tau_1 \int_0^{\tau_1} \dots \int_0^{\tau_{m-1}} f(\tau_m, .) \circ \dots \circ f(\tau_1, .) \mathrm{d}\tau_m$$

how to compute the chronological exponential series ?

how to express the F_i^δ's in terms of the f_i's ?

R1 *an ordinary exponential series solution :*

$$\overrightarrow{\exp} \int_0^\delta f(\tau, .) \mathrm{d}\tau = e^{\delta \mathcal{F}(f^\delta)} = F^\delta = 1 + \sum_{i \geq 1} \frac{\delta^i}{i!} F_i$$

the exponential Lie series solution

- F^δ is described by an ordinary exponential of operators :

from Integro/Differential (Poincaré) to Formal Lie series (Gröbner, 1973)

$$\overrightarrow{\exp} \int_0^\delta f(\tau,.)\mathrm{d}\tau \;=\; e^{\delta\mathcal{F}(f^\delta)} = F^\delta = 1 + \sum_{i\geq 1} \frac{\delta^i}{i!} F_i$$

$$\downarrow\uparrow$$

$$\delta\mathcal{F}(f^\delta) \;=\; \mathrm{Log}(1 + \sum_{i\geq 1} \frac{\delta^i}{i!} F_i)$$

- $\delta\mathcal{F}(f^\delta)$ admits a Lie series expansion : $\rightarrow$ Lie brackets of VF : $\mathrm{ad}_{f_i} f_j = [f_i, f_j]$

$$\delta\mathcal{F}(f^\delta) \;=\; \sum_{i\geq 1} \frac{\delta^i}{i!} B_i(f_0, \ldots, f_{i-1}) = \delta B_1 + \frac{\delta^2}{2!} B_2 + \frac{\delta^3}{3!} B_3 + \ldots$$

with: $B_1 = f_0$; $\quad B_2 = f_1 \quad B_3 = f_2 + 1/2\,\mathrm{ad}_{f_0} f_1$

$B_4 = f_3 + \mathrm{ad}_{f_0} f_2$; $\quad B_5 = f_4 + \frac{3}{2}\mathrm{ad}_{f_0} f_3 + \mathrm{ad}_{f_1} f_2 + \frac{1}{2}\mathrm{ad}^2_{f_1} f_0 + \frac{1}{6}\mathrm{ad}^2_{f_0} f_2 - \frac{1}{6}\mathrm{ad}^3_{f_0} f_1$

- Each B_i is a Lie polynomial of degree $i > 1$ in the f_j's :

$$B_{i+1} = f_i + \sum_{0\leq q<i} \sum_{\substack{l_1,\ldots,l_j\geq 1 \\ \Sigma_p(l_p)+q=i}} \frac{i!(-1)^j b_j}{j!q!l_1!\ldots l_j!} \mathrm{ad}_{B_{l_1}} \circ \ldots \circ \mathrm{ad}_{B_{l_j}} f_q.$$

exact and approximate sampled models

- exact sampled dynamics in the form of asymptotic series

$$1+\delta F_1+\frac{\delta^2}{2!}F_2+\frac{\delta^3}{3!}F_3+\ldots=e^{\delta\mathcal{F}(f^\delta)}=e^{\delta f_0+\frac{\delta^2}{2!}f_1+\frac{\delta^3}{3!}(f_2+1/2\mathrm{ad}_{f_0}f_1)+\ldots}$$

$\rightarrow$ be carefull ! $e^{\delta\mathcal{F}(f^\delta)}\neq e^{\int_0^\delta f(\tau,.)d\tau}$

$\rightarrow$ *conditions on the Lie algebra generated by the f_i directly modify the series exponent*

- *usual truncation :*

$$\begin{aligned}F_a^\delta &= 1+\delta F_1+\frac{\delta^2}{2!}F_2+\frac{\delta^3}{3!}F_3+O(\delta^4)\\ &= 1+\delta f_0+\frac{\delta^2}{2!}(f_0^2+f_1)+\frac{\delta^3}{3!}(f_0^3+2f_0\circ f_1+f_1\circ f_0+f_2)+O(\delta^4)\\ \tilde{F}_a^\delta &= 1+\delta f_0\end{aligned}$$

Euler scheme at the order 1 in δ

- *truncation of the exponent series :*

$$F_b^\delta = e^{\delta f_0+\frac{\delta^2}{2!}f_1+\frac{\delta^3}{3!}(f_2+1/2\mathrm{ad}_{f_0}f_1)+O(\delta^4)}$$

$$\tilde{F}_b^\delta = e^{\delta f_0+O(\delta^2)} \leftarrow \dot{x}(t)=f_0(x(t))$$

Euler type scheme

- *polynomial approximations in x*

why to look for an exponential form

$$\dot{x}(t) = f(t, x(t)) \longleftrightarrow x_{k+1} = e^{\delta \mathcal{F}_k^\delta}(x_k)$$

...

- exponential structure maintained for any mapping $y = h(x)$:

$$y_{k+1} = h(x_{k+1}) = e^{\delta \mathcal{F}_k^\delta} h|_{x_k}$$

- exponential suitable for composition over several steps :

$$x_{k+1} = e^{\delta \mathcal{F}_0^\delta} \circ \ldots \circ e^{\delta \mathcal{F}_k^\delta}(x_0)$$

...

Part 2 : Discrete-Time dynamics as coupled Differential/Difference equations - DDR

$$x_{k+1} = F(x_k, u_k)$$

Differential/Difference Representation DDR $(F_0, G(., u))$

ODE in u for the forced evolution :

$$\frac{dx^+(u)}{du} = G(x^+(u), u)$$

with initial condition given by the discrete mapping - jump:

$$x \to x^+(0) = F_0(x)$$

Key novel ideas:

- $x^+(u)$ *is a curve in* R^n *parameterized by* $u \in \mathcal{U}$

- $G(.,u)$: *analytic and complete VF on* R^n *parameterized by* u*, specifies the tangent VF of the curve* $x^+(u)$ *(continuous* u*-dependency); forced evolution imbedded into an ODE*

- F_0: *analytic on* R^n*; the free dynamics induces a jump*

- *Nonlinearity in* u *of the ODE:* $G(.,u) = G_1 + \sum_{i \geq 1} \frac{u^i}{i!} G_i$

$\longrightarrow$ *Hybrid state space representation of NL DT D!*

$\longrightarrow$ *Unified framework: Continouous-Time, Discrete-Time and Sampled versus Hybrid Dynamics*

How to compute the evolution ?

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \quad (1)$$

$$x^+(0) = F_0(x) \qquad \textit{(initial jump)} \quad (2)$$

step 0:initial conditions $(x_0, u_0, ..., u_p)$

step 1: compute $F(x_0)$ *according to (2) (jump) - integrate (1) between* 0 *and* u_0 *with initialization at* $x^+(0) = F(x_0)$ *to get*

$$x_1 = x^+(u_0)$$

step 2: compute $x^+(0) = F_0(x_1)$ *from* x_1 *according to (2), integrate (1) between* 0 *and* u_1 *with initialization at* $x^+(0) = F(x_1)$ *to get*

$$x_2 = x^+(u_0, u_1)$$

.. *step p: Iterate the construction to find for* $p \geq 3$

$$x_p = x^+(u_0, ..., u_{p-1})$$

About the existence of the DDR ?

- *Directly from modeling*

- *Under the existence of* $G(., u)$ *on* $R^n \times \mathcal{U}$ *s.t.*

$$G(x,u)|_{F(x,u)} = \frac{\partial F(x,u)}{\partial u} \qquad (A)$$ *satisfied (locally) for :*

→ *Drift invertible dynamics (equivalently there exists* u_0 *s.t.* $F(., u_0)$ *invertible)*

→ *Submersive dynamics (equivalent under feedback to drift invertible ones)*

- *Sampled dynamics OK :*

$$F(x,0) = e^{\delta f} x \qquad F^{-1}(x,0) = e^{-\delta f} x$$

Two Equivalent State-Space Representations of DT Dynamics

Given a mapping : $(x, u) \rightarrow F(x, u)$

derivation $\downarrow$ with respect to u

Let: $G(., u)$ *such that:* $G(., u)|_{F(x,u)} = \dfrac{\partial F(x, u)}{\partial u}$

(provided $F(x, u)$ *invertible (OK under sampling))*

$\uparrow\downarrow$

Reciprocally given a DDR :

$$\frac{dx^+(u)}{du} = G(x^+(u), u)$$

$$x^+(0) = F_0(x)$$

integration $\downarrow$ *with respect to* u

$$x^+(u) = \overrightarrow{\exp} \int_0^u G(x^+(0), v)dv \quad \textit{(chronological exponential)}$$

$$x^+(u) = x^+(0) + \sum_{m \geq 1} \int_0^u dv_1 ... \int_0^{v_{m-1}} dv_m G(., v_m) \circ ... \circ G(x^+(0), v_1)$$

$$F(x, u) = e^{u\mathcal{G}(.,u)} x|_{F_0(x)}$$

$u\mathcal{G}(., u)$: *explicitely computable Lie series* - (MNCC,2002)

More about integration

Theorem :

$$x^{+}(u) = \phi(u, 0, x^{+}(0)) = \Phi(u, 0)(x^{+}(0))$$

- $\Phi(u, 0)$:*(flow) admits an exponential representation:*

$$\Phi(u, 0) = e^{u\mathcal{G}(.,u)}$$

$$u\mathcal{G}(.,u) = \sum_{i\geq 1} \frac{u^i}{i!}\mathcal{B}_i = u\mathcal{B}_1 + \frac{u^2}{2}\mathcal{B}_2 + \frac{u^3}{3!}\mathcal{B}_3 + \frac{u^4}{4!}\mathcal{B}_4 + O(u^5)$$

- $\mathcal{B}_i$:*homogeneous Lie polynomial of degree i in the G_i*

$$\mathcal{B}_{i+1} = G_{i+1} + \sum_{0\leq k<i} \sum_{\substack{l_1,\ldots,l_j\geq 1 \\ \Sigma_p(l_p)+k=i}} \frac{(-1)^j i! b_j}{j! k! l_1! \ldots l_j!} ad_{\mathcal{B}_{l_1}} \circ \ldots \circ ad_{\mathcal{B}_{l_j}} G_{k+1}$$

$$\mathcal{B}_1 = G_1; \quad \mathcal{B}_2 = G_2; \quad \mathcal{B}_3 = G_3 + 1/2[G_1, G_2]; \quad \mathcal{B}_4 = G_4 + [G_1, G_3]$$

$$\begin{aligned} \mathcal{B}_5 &= G_5 + \frac{3}{2}[G_1, G_4] + [G_2, G_3] + \frac{1}{6}[G_1[G_1, G_3]] \\ &- \frac{1}{2}[G_2[G_1, G_2]] - \frac{1}{6}[G_1[G_1[G_1, G_2]]] \end{aligned}$$

$$\downarrow$$

Geometric differential technics and tools can be definitively used to handle NLDTD and thus NLSD

DDR under Sampling

DDR under Sampling

Theorem : *Given* $\dot{x}(t) = f(x(t)) + u_k g(x(t))$

There exists $(F_0^\delta, G^\delta(., u))$ *such that*

$$\begin{aligned} \frac{dx^+(u)}{du} &= G^\delta(x^+(u), u)) \\ x^+(0) &= F_0^\delta(x) = e^{\delta f} x \end{aligned}$$

$$G^\delta(., u) \;=\; \int_0^\delta e^{-s ad_{f+ug}} g ds = \delta g + \sum_{i \geq 1} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_{f+ug}^i g$$

$$G^\delta(., u) = G_1^\delta + \sum_{i \geq 1} \frac{u^i}{i!} G_i^\delta$$

(linearity in u lost under sampling)

Iterative computations of the G_i^δ: $G^\delta(.,u) = G_1^\delta + \sum_{i\geq 1} \frac{u^i}{i!} G_i^\delta$

$$G_1^\delta = \int_0^\delta e^{-s\,ad_f} g\,ds = \delta g + \sum_{i\geq 1} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^i g \qquad (u=0)$$

$$G_1^\delta = \delta g - \delta ad_f g + \frac{\delta^2}{2!} ad_f^2 g - \frac{\delta^3}{3!} ad_f^3 g + O(\delta^4)$$

$$G_1^\delta = \sum_{i\geq 0} \frac{\delta^{i+1}}{(i+1)!} X_i \in \mathcal{L}_0 \qquad \textit{(setting } X_i = (-1)^i ad_f^i g)$$

$$G_2^\delta = -\int_0^\delta ds_1 \int_0^{s_1} e^{-s\,ad_f} \uplus ad_g g\,ds \quad \in [\mathcal{L}_0, \mathcal{L}_0]$$

$$= \sum_{i\geq 2} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^{i-1} \uplus ad_g g$$

$\mathcal{L}_0$: *Lie ideal of* $\mathcal{L}(f,g)$ *generated by* g *(combinatoric proofs)*

Iterative computations of the G_i^δ: *(follow)*

$$
\begin{aligned}
G_{p+1}^{\delta} &= (-1)^p p! \int_0^\delta ds_p \int_0^{s_p} \dots \int_0^{s_1} e^{-s ad_f} ⧢ ad_g^p g ds \\
&= p! \sum_{i \geq p+1} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^{i-p} ⧢ ad_g^p g \qquad \in \mathcal{L}_0^{p+1}
\end{aligned}
$$

→ *Combinatorial proofs*

the shuffle product ⧢ *(integrations by parts) is defined in a recursive way:*

$$
1 ⧢ \zeta_i = \zeta_i ⧢ 1 = \zeta_i, \quad \zeta_i ⧢ \zeta_j = \zeta_j ⧢ \zeta_i = \zeta_i . \zeta_j + \zeta_j . \zeta_i
$$

$$
\begin{aligned}
\zeta_{i_1} \dots \zeta_{i_m} ⧢ \zeta_{j_1} \dots \zeta_{j_p} &:= \zeta_{i_1} (\zeta_{i_2} \dots \zeta_{i_m} ⧢ \zeta_{j_1} \dots \zeta_{j_p}) \\
&+ \zeta_{j_1} (\zeta_{i_1} \dots \zeta_{i_m} ⧢ \zeta_{j_2} \dots \zeta_{j_p})
\end{aligned}
$$

More about the need of formal calculus

The integro-differential method

$$\frac{\partial G^\delta(.,u)}{\partial u} = \int_0^\delta [\frac{\partial G^s(.,u)}{\partial s}, G^s(.,u)]ds$$

$$G_2^\delta = \int_0^\delta [\frac{\partial G_1^s}{\partial s}, G_1^s]ds = -G_1^\delta * G_1^\delta \qquad (u = 0)$$

$*$:*chronological product (non associative)*

$$G_2^\delta = \sum_{i \geq 2} \frac{\delta^{i+1}}{(i+1)!} \sum_{k=0}^{i-1} C_i^k [X_k, X_{i-k-1}] \qquad \textit{with} \qquad C_i^k := \frac{i!}{k!(i-k)!}$$

$$G_2^\delta = -\frac{\delta^3}{3!}[X_0, X_1] - \frac{2\delta^4}{4!}[X_0, X_2] - \frac{3\delta^5}{5!}[X_0, X_3] - \frac{2\delta^5}{5!}[X_1, X_2] + 0(\delta^6)$$

Iterately for the other vector fields :

$$G^{\delta}_{p+1} = (G^{\delta}_{p})^{(+)}$$

(the formal rule $(.)^{(+)}$ *"mimics" derivation with respect to* u*)*

$$G^{\delta}_{3} = (G^{\delta}_{2})^{(+)} = \int_0^{\delta} [\frac{\partial (G^s_1)^{(+)}}{\partial s}, G^s_1] ds + \int_0^{\delta} [\frac{\partial G^s_1}{\partial s}, (G^s_1)^{(+)}] ds$$

$$G^{\delta}_{3} = \int_0^{\delta} [\frac{\partial G^s_2}{\partial s}, G^s_1] ds + \int_0^{\delta} [\frac{\partial G^s_1}{\partial s}, G^s_2] ds = \frac{3}{2} G^{\delta}_1 * (G^{\delta}_1 * G^{\delta}_1)$$

$$G^{\delta}_{3} = \frac{2\delta^4}{4!}[X_0[X_0, X_1]] + \frac{6\delta^5}{5!}[X_0[X_0, X_2]] + \frac{2\delta^5}{5!}[X_1[X_0, X_1]] + 0(\delta^6)$$

A formal recursive law for computing the G_i^δ's

Starting from: $G_1^\delta = \sum_{i\geq 0} \frac{\delta^{i+1}}{(i+1)!} X_i \quad \in \mathcal{L}_0$

Setting: ${X_0}^+ = {X_1}^+ = 0;$

$${X_i}^+ := \sum_{k=0}^{i-1} C_i^k [X_k, X_{i-k-1}]$$

Compute iteratively :

$$\begin{aligned} G_2^\delta &= \sum_{i\geq 0} \frac{\delta^{i+1}}{(i+1)!} {X_i}^+ = \sum_{i\geq 2} \frac{\delta^{i+1}}{(i+1)!} \sum_{k=0}^{i-1} C_i^k [X_k, X_{i-k-1}] \quad \in [\mathcal{L}_0, \mathcal{L}_0] \\ &= -\frac{\delta^3}{3!}[X_0, X_1] - \frac{2\delta^4}{4!}[X_0, X_2] - \frac{3\delta^5}{5!}[X_0, X_3] - \frac{2\delta^5}{5!}[X_1, X_2] + 0(\delta^6) \end{aligned}$$

$$
\begin{aligned}
G_3^\delta &= \sum_{i\geq 2} \frac{\delta^{i+1}}{(i+1)!} X_i{}^{++} \\
&= \frac{2\delta^4}{4!}[X_0[X_0,X_1]] + \frac{6\delta^5}{5!}[X_0[X_0,X_2]] + \frac{2\delta^5}{5!}[X_1[X_0,X_1]] + 0(\delta^6) \\
G_4^\delta &= \sum_{i\geq 2} \frac{\delta^{i+1}}{(i+1)!} X_i{}^{+++} \\
&= -\frac{6\delta^5}{5!}[X_0[X_0,[X_0,X_1]]] + 0(\delta^6) \\
&\cdots
\end{aligned}
$$

$\rightarrow$ *algorithmic procedure for computing exact models*

$\rightarrow$ *approximated solutions computed through series truncations are sufficient in practice*

some impact in analysis and control design

A link between the map $e^{\delta f+\delta ug}$ and the DDR

Given :

$$\frac{dx^+(u)}{du} = G^\delta(x^+(u),u))$$

$$x^+(0) = e^{\delta f}x$$

under integration $\downarrow$ *with respect to u*

$$x^+(u) = e^{u\mathcal{G}^\delta(.,u)}(x^+(0))$$

with series exponent : $u\mathcal{G}^\delta(.,u) = \sum_{p\geq 1}\frac{u^p}{p!}\mathcal{B}_p^\delta(G_1^\delta,...,G_p^\delta)$

$\mathcal{B}_p^\delta(G_1^\delta,\ldots,G_p^\delta)$: *homogeneous Lie polynomial of degree* p.

$$\mathcal{B}_1^\delta = G_1^\delta; \quad \mathcal{B}_2^\delta = G_2^\delta; \quad \mathcal{B}_3^\delta = G_3^\delta + 1/2[G_1^\delta, G_2^\delta]; \quad \mathcal{B}_4^\delta = G_4^\delta + [G_1^\delta, G_3^\delta]$$

$$\begin{aligned} \mathcal{B}_5^\delta &= G_5^\delta + \frac{3}{2}[G_1^\delta, G_4^\delta] + [G_2^\delta, G_3^\delta] + \frac{1}{6}[G_1^\delta[G_1^\delta, G_3^\delta]] \\ &- \frac{1}{2}[G_2^\delta[G_1^\delta, G_2^\delta]] - \frac{1}{6}[G_1^\delta[G_1^\delta[G_1^\delta, G_2^\delta]]] \end{aligned}$$

$$\begin{aligned} x^+(u) &= e^{\delta f} \circ e^{u\mathcal{G}^\delta(.,u)} x = e^{\delta f + \delta u g} x \\ &= e^{\delta f} \circ e^{uG_1^\delta + \frac{u^2}{2}G_2^\delta + \frac{u^3}{3!}(G_3^\delta + 1/2[G_1^\delta, G_2^\delta]) + O(u^4)} x \end{aligned}$$

Decomposition of the dynamics around the free evolution

The behaviour over several time-steps as Multi-Input SD

Given : $x^+(u_0, u_1) = e^{\delta f + u_0 \delta g} \circ e^{\delta f + u_1 \delta g} x_0$, *the DDR is :*

$$
\begin{aligned}
\frac{\partial x^+(u_0, u_1)}{\partial u_0} &= G^\delta_{u_1}(x^+(u_0, u_1), u_0) \\
\frac{\partial x^+(u_0, u_1)}{\partial u_1} &= G^\delta(x^+(u_0, u_1), u_1) \\
x^+(0) &= e^{2\delta f} x_0
\end{aligned}
$$

$\rightarrow$ *A set of forced partial derivatives equations*

with $G^\delta_{u_1}(., u_0)$: *transport of* $G^\delta(., u_0)$ *along* $e^{\delta f + \delta u_1 g}$

idem $Ad^{p\delta f}_{G_i}$: *transport of* G^δ_i *along* $e^{p\delta f}$ *MNC (1984), Jakubczyk-NC(1984), Sontag Albertini(1990)*

In this sampled context, we get :

$$\begin{aligned} G^{\delta}_{u_1}(.,u_0) &:= e^{-ad_{\delta f+\delta u_1 g}} G^{\delta}(\cdot,u_0) \\ Ad^{p\delta f}_{G_i} &:= e^{-ad_{p\delta f}} G^{\delta}_i = e^{-p\delta f} \circ G^{\delta}_i \circ e^{p\delta f} \\ &= G_i^{(p+1)\delta} - G_i^{p\delta} \end{aligned}$$

- *under successive compositions (1^{st} order approximations in u of the series exponent):*

$$x_p = e^{p\delta f} \circ e^{u_0 Ad^p_{\delta f} G^{\delta}_1} \circ \ldots \circ e^{u_p G^{\delta}_1} x_0$$

$\mathcal{L}(G_1^{\delta}, ..., Ad_{G_1}^{p\delta f}, ...)$: *defines the Lie algebra of controllability*

$$\begin{pmatrix} G_1^{\delta} \\ Ad_{G_1}^{\delta f} \\ ... \\ Ad_{G_1}^{(p-1)\delta f} \end{pmatrix} = \begin{pmatrix} \delta & \delta^2/2! & ... & \delta^p/p! \\ 2\delta & (2\delta)^2/2! & ... & (2\delta)^p/p! \\ ... & ... & ... & ... \\ p\delta & (p\delta)^2/2! & ... & (p\delta)^p/p! \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ ... \\ X_{p-1} \end{pmatrix}$$

when $X_i = (-1)^i ad_f^i g$

A fundamental tool to study

- Local Controllability/Observability criteria $\rightarrow$ State-Space decomposition
- Characterization of Invariant Structures, Parameterized Canonical forms, Passivity or Losslessness
- Control Geometric Properties as Feedback Linearizability, Decoupling with Stability under Static/Dynamic feedback, Regulation, Disturbance Rejection
- Homological-like Equations $\rightarrow$ Approximated Normal Forms and approximated Feedback Linearizability
- Strong parallelism between discrete-time dynamics

$$\frac{dx^+(u)}{du} = G_1(x^+(u)) \qquad\qquad x^+(0) = F_0(x)$$

  and input-affine continuous-time dynamics

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t))$$

About some extensions

About some extensions

- Nonlinearity in u : $\dot{x}(t) = f(x(t), u_k)$ *(ZOH)*

$$G^{\delta}(., u_k) = \int_0^{\delta} e^{-s ad_{f(.,u_k)}} \frac{\partial f(., u_k)}{\partial u_k} ds$$

- Multi-input case : $\dot{x}(t) = f(x(t), u_{1k}, \ldots, u_{mk})$

$$_mG^{\delta}(., \underline{u}_k) = \int_0^{\delta} e^{-s ad_{f(.,\underline{u}_k))}} \frac{\partial f(., \underline{u}_k)}{\partial u_{mk}} ds$$

About some extensions(*follow*)

- Time-varying case : $\dot{x}(t) = f(x(t), t, u_k)$

$$G^\delta(., u_k) = \int_0^\delta e^{-s ad_{\mathcal{F}(., s, u_k)}} \frac{\partial f(., s, u_k)}{\partial u_k} ds$$

with : $x(\delta) = e^{\delta \mathcal{F}(., \delta, u_k)} x_0$

closed-loop, parameter dependent or perturbed dynamics, ...

- Higher Order Holding Device $\rightarrow$ Multi-Input SD

$$\dot{x}(t) = f(x(t), u_k + t\dot{u}_k) \quad \dot{u}_k := \frac{du(t)}{dt}|_{t=k\delta}$$ 1*st order device*

$\rightarrow$ *The sampled dynamics depends on* u_k *and* $\dot{u}_k$

$\rightarrow$ *Digital design over* u_k *and* $\dot{u}_k$ *(piecewise discontinuous control)*

Summaries : Sampling is useful :

- *To discuss how structural and control properties are (can be) maintained under sampling*
- *To design multirate control strategies for maintaining under sampling control properties or solving difficult problems (no standard smooth continuous-time design) ...*
- *To set state space representation of hybrid dynamics*
- *Favorite examples: motion planning or articulated mechanical structures (nonholonomic constraints) and walking robots (switching dynamics)*
- *Computational aspects* $\rightarrow$ *Approximated models through series truncations*

Some Examples

Chained dynamics under sampling (ex. on R^3)

$$\dot{x}_1(t) = u_1 \qquad \dot{x}_2(t) = u_2 \qquad \dot{x}_3(t) = x_2 u_1$$

$\mathcal{L}(g_1, g_2)$ *non involutive but* $\rho(g_1, g_2, [g_1, g_2]) = 3$

Exact Driftless sampled equivalent $(x_i^+(0) = x_i)$

$$\begin{pmatrix} \frac{dx_1^+(u)}{du_1} \\ \frac{dx_2^+(u)}{du_1} \\ \frac{dx_3^+(u)}{du_1} \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \\ \delta x_2^+(u) - \frac{\delta^2 u_2}{2} \end{pmatrix} ; \begin{pmatrix} \frac{dx_1^+(u)}{du_2} \\ \frac{dx_2^+(u)}{du_2} \\ \frac{dx_3^+(u)}{du_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \\ \frac{\delta^2 u_1}{2} \end{pmatrix}$$

with partial derivatives forced equations

$$G_1^\delta(\cdot,u) = (\delta, 0, \delta x_2 - \frac{\delta^2}{2}u_2)^T \qquad G_2^\delta(\cdot,u) = (0, \delta, \frac{\delta^2 u_1}{2})^T$$

$$\downarrow$$

$$\rho(G_1^\delta(\cdot,u), G_2^\delta(\cdot,u), [G_1^\delta(\cdot,u), G_2^\delta(\cdot,u)]) = 3$$

$$\downarrow$$

Full inversion under multirate of order 2

$\rightarrow$ *exact state steering over 2 time steps*

Idem for various extensions of chained/driftless dynamics

$\rightarrow$ *exact finite sampling*

Polynomial Normal Forms (ex. quadratic on R^3)

$$\dot{x_1}(t) = x_2 + a x_3^2 \qquad \dot{x_2}(t) = x_3 \qquad \dot{x_3}(t) = u$$

$$X_0 = g = (0,0,1)^T; \qquad X_1 = -ad_f g = (2ax_3, 1, 0)^T$$

$$X_2 = ad_f^2 g = (1,0,0)^T; \qquad [X_0, X_1] = (0,0,2a)^T; \qquad X_{i\geq 3} = 0$$

Sampled equivalent:

$$\begin{pmatrix} \frac{dx_1^+(u)}{du} \\ \frac{dx_2^+(u)}{du} \\ \frac{dx_3^+(u)}{du} \end{pmatrix} = \begin{pmatrix} \delta^2 a x_3^+(u) + \frac{\delta^3}{3!} \\ \frac{\delta^2}{2!} \\ \delta \end{pmatrix} - u \begin{pmatrix} \frac{\delta^3 a}{3} \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \end{pmatrix} = \begin{pmatrix} x_1 + \delta x_2 + \delta a x_3^2 + \frac{\delta^2 x_3}{2} \\ x_2 + \delta x_3 \\ x_3 \end{pmatrix}$$

→*Exact sampled equivalent of finite degree in u and δ*

→*Quadraticity in (x, u) is maintained*

$$\begin{aligned} G_1^{\delta}(x) &= \delta X_0 + \frac{\delta^2}{2!}X_1 + \frac{\delta^3}{3!}X_2 \\ G_2^{\delta}(x) &= -\frac{\delta^3}{3!}[X_0, X_1]; \qquad G_{i\geq 3}^{\delta}(x) = 0 \\ Ad_{G_1}^{\delta f}(x) &= \delta X_0 + \frac{3\delta^2}{2!}X_1 + \frac{7\delta^3}{3!}X_2 = G_1^{2\delta}(x) - G_1^{\delta}(x) \\ Ad_{G_1}^{2\delta f}(x) &= 2\delta X_0 + \frac{3\delta^2}{2!}X_1 + \frac{7\delta^3}{3!}X_2 = G_1^{3\delta}(x) - G_1^{2\delta}(x) \end{aligned}$$