Converse Lyapunov methods for switched systems

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Switched systems

A switched system is something of the type

$$\dot{x} = f_{\sigma(t)}(x), \quad x \in \mathbf{R}^d$$

$\sigma : \mathbf{R} \to \overline{\Sigma}$ is a function (unknown) taken in a class Σ (known). Assume that $f_{\bar{\sigma}}(0) = 0$ for every $\bar{\sigma}$ and consider the problem of global uniform asymptotic stability (GUAS) of 0:

> *Does* $\beta \in \mathcal{KL}$ *exists such that* $|x(t; \sigma(\cdot), x_0)| \leq \beta(|x_0|, t)$ $\forall t \geq 0$, *uniformly with respect to* $\sigma \in \Sigma$*?*

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- $\overline{\Sigma}$ finite, $\sigma(\cdot)$ piecewise constant or measurable, $f_{\bar{\sigma}}$ linear vector field for every $\bar{\sigma}$. In this case GUAS is equivalent to global uniform exponential stability (GUES) ($\beta(|x_0|, t) = Ke^{-\mu t}|x_0|$ for some $\mu > 0$)

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- similar definitions in the discrete-time or hybrid-time settings
- $\Sigma = \Sigma_\tau = \{\sigma(\cdot)$ piecewise constant $|$ discontinuities separated by at least $\tau\}$, $\tau > 0$ dwell time

Direct Lyapunov method

Direct Lyapunov Theorem

If there exist a positive definite $V : \mathbf{R}^d \to [0, +\infty)$ of class C^1 and $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{PD}$ such that

- $V(x) \to +\infty$ as $|x| \to +\infty$
- $\alpha_0(|x|) \leq V(x) \leq \alpha_1(|x|)$
- $\nabla V(x) \cdot f_{\bar{\sigma}}(x) \leq -\alpha_2(|x|)$

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The converse (stability implies existence of Lyapunov function) is clearly not true without some further assumption on the class of admissible signals Σ.

The linear case with CQLF

Let $f_{\bar{\sigma}} x = A_{\bar{\sigma}} x$ for every $\bar{\sigma}$ and x.
If V is looked for in the form $V(x) = x^T P x$ with $P > 0$, then the conditions which P should satisfy to fit the hypothesis of the Direct Lyapunov Theorem are

$$A_{\bar{\sigma}}^T P + P A_{\bar{\sigma}} < 0 \quad \forall \bar{\sigma}$$

We speak of common quadratic Lyapunov function (CQLF).

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By changing the form of the Lyapunov function to $V(x_n, \sigma_n) = x_n^T P_{\sigma_n} x_n$, we get the conditions

$$A_{\bar{\sigma}}^T P_{\hat{\sigma}} A_{\bar{\sigma}} - P_{\bar{\sigma}} < 0 \quad \forall \bar{\sigma}, \hat{\sigma}$$

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All these conditions are linear matrix inequalities (LMI) for which powerful numerical resolution methods exist.

Converse Lyapunov method

Consider now the case where $\overline{\Sigma}$ is finite and Σ is made of all piecewise constant (or measurable) functions with values in $\overline{\Sigma}$.

Converse Lyapunov theorem

If the system is GUES then there exists a C^∞ positive definite function V and $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{PD}$ such that

- $V(x) \to +\infty$ as $|x| \to +\infty$
- $\alpha_0(|x|) \leq V(x) \leq \alpha_1(|x|)$
- $\nabla V(x) \cdot f_{\bar{\sigma}}(x) \leq -\alpha_2(|x|)$

Converse Lyapunov theorems date back at least to [Massera, Annals of Mathematics, 1949].
The first nonlinear switched system formulations are due to [Lin, Sontag, Wang, 1996], [Dayawansa, Martin, 1999], [Mancilla-Aguilar, García, 2000].

The linear case

Let us consider the linear case $\dot{x} = A_{\sigma(t)}x, \quad \sigma \in \Sigma$.
An interesting question is that of universal classes of Lyapunov functions, i.e., families $\mathcal{V}$ of positive definite, unbounded at infinity functions such that GUES is equivalent to the existence of a Lyapunov function in $\mathcal{V}$.

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- the class of polyhedral Lyapunov functions is universal [Molchanov, Pyatniskiy, 1986]. Generalized in [Blanchini, Miani, 1996] to $V(x) = \|(v_1 \cdot x, \ldots, v_r \cdot x)\|_p$

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- the class of polynomial Lyapunov functions is universal [Dayawansa, Martin, 1999]
- there exist no $M \in \mathbf{N}$ such that the class of polynomial Lyapunov functions of degree $\leq M$ is universal [Mason, Boscain, Chitour, 2006]

Construction of Lyapunov functions

The converse Lyapunov function is usually defined taking, in a first step,

$$V(x_0) = \sup_{\sigma(\cdot)\in\Sigma} \int_0^\infty \|x(t;\sigma(\cdot),x_0)\|^2 dt.$$

An alternative choice is

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- regularity? In general difficult to assess from the definition.

Smoothing of the Lyapunov function

It is easy from the definition of V to get that, for every $\bar{\sigma} \in \overline{\Sigma}$ and $x \in \mathbf{R}^d$,

$$\limsup_{t \to 0^+} \frac{V(e^{tA_{\bar{\sigma}}} x) - V(x)}{t} \leq -|x|^2.$$

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The level sets of the Lyapunov function are homogeneous. By modifying them slightly one can obtain a smooth Lyapunov function (convolution) or a polyhedral one (rectification) or ....

Weak Lyapunov functions in the marginally stable case

The maximal Lyapunov exponent is defined as

$$\rho(\Sigma) = \sup_{\sigma(\cdot),\, x_0} \left(\limsup_{t\to\infty} \frac{1}{t} \log \|x(t;\sigma(\cdot),x_0)\| \right).$$

If $\overline{\Sigma} = \{\bar{\sigma}\}$, $\rho(\Sigma) = \max_{\lambda \in \mathrm{spectrum}(A_{\bar{\sigma}})} \Re(\lambda)$.

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$\rho(\Sigma) < 0$

GUES

$\rho(\Sigma) = 0$

- **marginally stable:** all trajectories are bounded and $\exists$ trajectory $\not\to 0$
- **marginally unstable:** $\exists$ unbounded trajectory

$\rho(\Sigma) > 0$

unstable: $\exists$ trajectory $\to \infty$ exponentially.

Irreducibility and Barabanov norms

Let $\mathcal{A} = \{A_{\bar\sigma} \mid \bar\sigma \in \overline{\Sigma}\}$.

Definition

$\mathcal{A}$ reducible if $\exists\ 0 \subsetneq V \subsetneq \mathbf{R}^d$ invariant for every $A \in \mathcal{A}$.
$\mathcal{A}$ irreducible otherwise.

$$\mathcal{A} \text{ reducible} \quad \Longleftrightarrow \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \ \forall A \in \mathcal{A}$$

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$\rho(\Sigma) = 0$ and $\mathcal{A}$ irreducible

$$\forall x_0 \in \mathbf{R}^d, \qquad v(x_0) := \sup_{\sigma(\cdot)} \limsup_{t\to+\infty} \|x\big(t;\sigma(\cdot),x_0\big)\|.$$

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Theorem (Barabanov, 1988)

$\rho(\Sigma) = 0$ and $\mathcal{A}$ irreducible. Then the system is marginally stable, $v : \mathbf{R}^d \to [0, +\infty)$ is a norm and

- $v(x(t; \sigma(\cdot), x_0)) \leq v(x_0)$ for every $\sigma \in \Sigma$;
- $\forall x_0$, $\exists\ \sigma \in \Sigma$ with $v(x(t; \sigma(\cdot), x_0)) \equiv v(x_0)$.

Infinite-dimensional case

Consider $\dot{x} = A_\sigma x$ in a Banach space X. The operators A_σ might be unbounded and have different domains.
A reasonable definition of solution for such a switched system is:

Definition

Assume that each $A_{\bar{\sigma}}$ generates a strongly continuous semigroup $T_{\bar{\sigma}}(\cdot)$, i.e., $T_{\bar{\sigma}}(t) \in L(X)\ \forall t \geq 0$; $T_{\bar{\sigma}}(0) = I_X$; $T_{\bar{\sigma}}(s+t) = T_{\bar{\sigma}}(s) \circ T_{\bar{\sigma}}(t)\ \forall s, t \geq 0$; $\lim_{t\to 0^+} T_{\bar{\sigma}}(t)x = x\ \forall x \in X$; $Ax = \lim_{t\to 0^+} \frac{T_{\bar{\sigma}}(t)x - x}{t}\ \forall x \in D(A_{\bar{\sigma}})$.

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For $\sigma : [0, \infty) \to \overline{\Sigma}$ piecewise constant left-continuous with switching times $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and for $t \in [t_k, t_k + 1)$, let

$$x(t; \sigma(\cdot), x_0) = T_{\sigma(t_k)}(t - t_k) \circ T_{\sigma(t_{k-1})}(t_k - t_{k-1}) \circ \cdots \circ T_{\sigma(t_0)}(t_1 - t_0)x_0.$$

Here $\overline{\Sigma}$ may be infinite.

Converse Lyapunov theorem in Banach space [Hante, Sigalotti, 2011]

The following three conditions are equivalent:

(A) GUES

(B) There exist $M \geq 1$ and $\omega > 0$ such that, $\forall \sigma(\cdot)$, $\forall x_0$,

$$\|x(t; \sigma(\cdot), x_0)\|_X \leq M e^{\omega t} \|x_0\|_X, \quad t \geq 0,$$

and there exists $V : X \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm in X,

$$V(x) \leq C \|x\|_X^2, \quad x \in X$$

$$\liminf_{t \to 0^+} \frac{V(T_{\bar{\sigma}}(t)x) - V(x)}{t} \leq -\|x\|_X^2, \quad \bar{\sigma} \in \overline{\Sigma},\ x \in X.$$

(C) There exists $V : X \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on X,

$$c\|x\|_X^2 \leq V(x) \leq C \|x\|_X^2, \quad x \in X$$

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A key lemma

Triggiani 1994, Hante, Sigalotti, 2011

Assume that

(a) there exist $M \geq 1$ and $w > 0$ such that $\forall \sigma(\cdot)$, $\forall x_0$,

$$\|T_{\sigma(\cdot)}(t)x_0 = x(t;\sigma(\cdot),x_0)\|_X \leq Me^{\omega t}\|x_0\|_X, \quad t \geq 0,$$

(b) there exist $c \geq 0$ and $p \in [1,+\infty)$ such that

$$\int_0^{+\infty} \|T_{\sigma(\cdot)}(t)\psi\|_X^p \leq c\|\psi\|_X^p,$$

for every $\psi \in X$ and every $\sigma(\cdot) \in \overline{\Sigma}$.

Then there exist $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0,\ \sigma(\cdot) \in \overline{\Sigma}.$$

Remarks

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- (B) and (C) are better suited test GUES and to deduce from GUES, respectively

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- In the special case in which X is a separable Hilbert space, we can prove the Fréchet directional differentiability of V and we establish a characterization of the directional Fréchet derivatives.
- smoothing is much harder in infinite dimension, even when $\overline{\Sigma}$ is finite. At least in the Hilbert case, one would like to prove that the class of functions of the form $V(x) = \|(v_1 \cdot x, \ldots, v_r \cdot x)\|_p$ is universal. This is still an open problem.

Retarded systems

Consider

$$\dot{x}(t) = \sum_{i=1}^{p} A_{\sigma(t),i}\, x\Big(t - \tau_i\big(\sigma(t)\big)\Big), \quad x(t) \in \mathbf{R}^d$$

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or, more generally,

$$\dot{x} = \Gamma(t) x_t$$

where $x_t : [-r, 0] \to \mathbb{R}^d$ is the history function $x_t(\theta) = x(t+\theta)$, and $\Gamma(\cdot)$ is piecewise constant or measurable in a set Q of operators

$$L : C([-r,0], \mathbb{R}^d) \to \mathbf{R}^d \quad \text{or} \quad L : H^1([-r,0], \mathbb{R}^d) \to \mathbf{R}^d.$$

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Such a system can be seen as a switching system in the Banach space $C([-r, 0], \mathbb{R}^d)$ or $H^1([-r, 0], \mathbb{R}^d)$.

Converse Lyapunov theorem for retarded systems [Haidar, Mason, Sigalotti, ≥2015]

Let $Q \subset \mathcal{L}(C([-r,0],\mathbb{R}^d),\mathbb{R}^d)$ be bounded. The following statements are equivalent:

(i) The system is GUES in $C([-r,0],\mathbb{R}^d)$.

(ii) The system is GUES in $H^1([-r,0],\mathbb{R}^d)$.

(iii) There exists $V : C([-r,0],\mathbb{R}^d) \to [0,\infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $C([-r,0],\mathbb{R}^d)$,

$$\underline{c}\|\psi\|_C^2 \le V(\psi) \le \bar{c}\|\psi\|_C^2$$

$$\limsup_{t\to 0^+} \frac{V(T_{\bar{\sigma}}(t)\psi) - V(\psi)}{t} \le -\|\psi\|_C^2, \quad \bar{\sigma} \in \overline{\Sigma}, \psi \in C([-r,0],\mathbb{R}^d).$$

(iv) There exists a directionally Fréchet differentiable function $V : H^1([-r,0],\mathbb{R}^d) \to [0,\infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $H^1([-r,0],\mathbb{R}^d)$,

$$\underline{c}\|\psi\|_{H^1}^2 \le V(\psi) \le \bar{c}\|\psi\|_{H^1}^2$$

$$\limsup_{t\to 0^+} \frac{V(T_{\bar{\sigma}}(t)\psi) - V(\psi)}{t} \le -\|\psi\|_{H^1}^2, \quad \psi \in H^1([-r,0],\mathbb{R}^d).$$

Converse Lyapunov theorem for retarded systems [Haidar, Mason, Sigalotti, ≥2015]

(v) There exists a continuous function

$$V : C([-r,0],\mathbb{R}^d) \to [0,\infty)$$

such that

$$V(\psi) \le c\|\psi\|_C^2$$

$$\liminf_{t\to 0^+} \frac{V(T_{\bar\sigma}(t)\psi) - V(\psi)}{t} \le -|\psi(0)|^2, \quad \psi \in C([-r,0],\mathbb{R}^d).$$

(vi) There exists a continuous function

$$V : H^1([-r,0],\mathbb{R}^d) \to [0,\infty)$$

such that

$$V(\psi) \le c\|\psi\|_{H^1}^2$$

for some constant $c > 0$ and

$$\liminf_{t\to 0^+} \frac{V(T_{\bar\sigma}(t)\psi) - V(\psi)}{t} \le -|\psi(0)|^2, \quad \psi \in H^1([-r,0],\mathbb{R}^d).$$