Convex conditions for the stability analysis and control of linear aperiodic impulsive systems with applications

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Introduction

Stability of impulsive systems

Stabilization of impulsive systems

Applications

Conclusion
Impulsive systems

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Applications

Conclusion

Impulsive systems

**Linear case**

\[
\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}_0} \\
x(t) &= Jx(t^-), \quad t \in \{t_k\}_{k \in \mathbb{N}_0} \\
x(0) &= x_0
\end{aligned}
\]  

where \( x(t^-) = \lim_{s \uparrow t} x(s) \).

- A continuous part
- A discrete part
- A set of impulse instants \( \{t_k\}_{k \in \mathbb{N}_0}, t_0 = 0 \), at which the jump rule applies

**Jumping rule**

- Time-dependent jumping instants (external)
- State-dependent jumping instants, e.g. when \( x \) enters some sets (internal)
• Stability depends on the matrices of the system but also on the sequence of impulse instants \( \{t_k\}_{k \in \mathbb{N}_0} \).

• Here the system is stable when \( t_{k+1} - t_k = 0.3 \) and unstable when \( t_{k+1} - t_k = 0.6 \).
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• How can we characterize stability in an efficient/accurate/tractable way?
- Stability depends on the matrices of the system but also on the sequence of impulse instants \( \{t_k\}_{k \in \mathbb{N}_0} \).
- Here the system is stable when \( t_{k+1} - t_k = 0.3 \) and unstable when \( t_{k+1} - t_k = 0.6 \).

- How can we characterize stability in an efficient/accurate/tractable way?
- How can we derive tractable conditions for control design?
Stability of impulsive systems

Introduction

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Stability analysis and stabilization of linear aperiodic impulsive systems
Dwell-times

Definition
The dwell-time $T_k$ is defined as $T_k = t_{k+1} - t_k$, i.e. $t_{k+1} = t_k + T_k$. 


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Average dwell-time

1. The average number of impulses in any time interval
2. Asymptotic notion

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Average dwell-time

- The average number of impulses in any time interval
- Asymptotic notion

Minimum/maximum/range dwell-time

- Minimum dwell-time: $T_k \geq \bar{T}$, for some $\bar{T} > 0$, $k \in \mathbb{N}_0$
- Maximum dwell-time: $T_k \leq \bar{T}$, for some $\bar{T} > 0$, $k \in \mathbb{N}_0$
- Range dwell-time: $T_k \in [T_{min}, T_{max}]$, for some $0 < T_{min} \leq T_{max} < \infty$, $k \in \mathbb{N}_0$

---

Theorem \(^{(1)}\)

Assume that there exist \(P \in \mathbb{S}^n_\succ 0\) and scalars \(c > 0, d < 0\), such that

\[
A^T P + PA + cP \prec 0
\]
\[
J^T P J - e^{-d} P \prec 0.
\]

Then, the system is stable provided that the number of impulses \(N(t, s)\) over the interval \((s, t]\) satisfies

\[
N(t, s) \leq \frac{t - s}{\tau^*} + N_0, \quad \text{for all } t \geq s.
\]
Theorem (1)

Assume that there exist $P \in \mathbb{S}_n^+ > 0$ and a scalar $\bar{T} > 0$ such that the conditions

$$A^T P + PA \prec 0$$
$$J^T e^{A^T \bar{T}} P e^{A\bar{T}} J - P \prec 0$$

hold.

Then, the system is stable provided that $T_k \geq \bar{T}$; i.e. $t_{k+1} \geq t_k + \bar{T}$, $k \in \mathbb{N}_0$. 

\[\text{REFERENCES}\]

Theorem (1)
Assume that there exist $P \in \mathbb{S}_+^n$ and a scalar $\bar{T} > 0$ such that the conditions

$$
A^T P + PA < 0
$$

$$
J^T e^{A^T \bar{T}} Pe^{A \bar{T}} J - P < 0
$$

hold.
Then, the system is stable provided that $T_k \geq \bar{T}$; i.e. $t_{k+1} \geq t_k + \bar{T}$, $k \in \mathbb{N}_0$.

Remark

- $A$ must be Hurwitz
- Stable continuous-time dynamics, potentially unstable discrete-time dynamics
- If we let $\bar{T} \to 0$, then we obtain a condition for arbitrary impulse times (but we must deal with Zeno behavior)
- Easy to check

---

Discrete-time system

We can associate the following discrete-time system with the initial impulsive system

\[ x(t_{k+1}^-) = e^{AT_k} J x(t_k^-), \quad k \in \mathbb{N}_0 \]  

(4)

where \( t_0 = 0 \) and \( T_k \in [T_{min}, T_{max}] \).
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Theorem (1)
Assume that there exist \( P \in \mathbb{S}_>^n \) such that the condition

\[ J^T e^{A^T \theta} P e^{A \theta} J - P < 0 \]  

(5)

holds for all \( \theta \in [T_{min}, T_{max}] \).

Then, the system is stable provided that \( T_k \in [T_{min}, T_{max}], k \in \mathbb{N}_0 \).
Discrete-time system

We can associate the following discrete-time system with the initial impulsive system

\[ x(t_{k+1}^-) = e^{AT_k}Jx(t_k^-), \quad k \in \mathbb{N}_0 \]  

where \( t_0 = 0 \) and \( T_k \in [T_{min}, T_{max}] \).

Theorem (1)

Assume that there exist \( P \in \mathbb{S}^n_{>0} \) such that the condition

\[ J^T e^{A^T \theta} P e^{A \theta} J - P \prec 0 \]  

holds for all \( \theta \in [T_{min}, T_{max}] \).

Then, the system is stable provided that \( T_k \in [T_{min}, T_{max}], \ k \in \mathbb{N}_0 \).

Remarks

- Robust feasibility problem (due to parametric dependence)
- Not easy to check since non-convex in \( \theta \) . . .
Difficulties & Proposed Solution

Difficulties

- Parameter dependence is at the exponential (not convenient)

\[ J^T e^{A^T \theta} P e^{A\theta} J - P \prec 0, \quad \theta \in [T_{min}, T_{max}] \]

- Difficult to extend to uncertain systems

\[ J^T e^{(A+\Delta)^T T} P e^{(A+\Delta)T} J - P \prec 0 \]

- Control design is non-convex

\[ J^T e^{(A+BK)^T T} P e^{(A+BK)T} J - P \prec 0 \]
Difficulties & Proposed Solution

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- Parameter dependence is at the exponential (not convenient)
  \[ J^T e^{A^T \theta} P e^{A \theta} J - P < 0, \quad \theta \in [T_{min}, T_{max}] \]

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  \[ J^T e^{(A+\Delta)^T \bar{T}} P e^{(A+\Delta)\bar{T}} J - P < 0 \]

- Control design is non-convex
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Getting rid of exponential terms

- Looped-functionals\(^a,\(^b\)
- Clock-dependent Lyapunov functions\(^c\)

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\(^c\) C. Briat. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints, *Automatica*, 2013
Convex conditions for periodic impulses

Theorem

Let us consider an impulsive system \((A, J)\) with periodic impulses, i.e. \(T_k = \bar{T}, k \in \mathbb{N}\). Then, the following statements are equivalent:

(a) The impulsive system with \(\bar{T}\)-periodic impulses is asymptotically stable.
Theorem

Let us consider an impulsive system \((A, J)\) with periodic impulses, i.e. \(T_k = \bar{T}, k \in \mathbb{N}\). Then, the following statements are equivalent:

(a) The impulsive system with \(\bar{T}\)-periodic impulses is asymptotically stable.

(b) There exists a matrix \(P \in \mathbb{S}^n_{>0}\) such that the LMI

\[
J^T e^{AT \bar{T}} P e^{AT} J - P \prec 0
\]  

holds.
Theorem

Let us consider an impulsive system \((A, J)\) with periodic impulses, i.e. \(T_k = \bar{T}, k \in \mathbb{N}\). Then, the following statements are equivalent:

(a) The impulsive system with \(\bar{T}\)-periodic impulses is asymptotically stable.

(b) There exists a matrix \(P \in \mathbb{S}_n^+\) such that the LMI

\[
J^T e^{A^T \bar{T}} P e^{A \bar{T}} J - P \prec 0
\]

holds.

(c) There exist a differentiable matrix function \(R : [0, \bar{T}] \rightarrow \mathbb{S}^n\), \(R(\bar{T}) \succ 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T R(\tau) + R(\tau) A + \dot{R}(\tau) \leq 0 \quad \text{and} \quad J^T R(0) J - R(\bar{T}) + \varepsilon I \leq 0
\]

hold for all \(\tau \in [0, \bar{T}]\).
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\]

hold for all \(\tau \in [0, \bar{T}]\).

(d) There exist a differentiable matrix function \(S : [0, \bar{T}] \mapsto \mathbb{S}_n^+, S(0) \succ 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T S(\tau) + S(\tau) A - \dot{S}(\tau) \leq 0 \quad \text{and} \quad J^T S(\bar{T}) J - S(0) + \varepsilon I \leq 0
\]

hold for all \(\tau \in [0, \bar{T}]\).
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Let us consider an impulsive system \((A, J)\). Then, the following statements are equivalent:

(a) There exists a matrix \(P \in \mathbb{S}_n^{>0}\) such that the LMI

\[
J^T e^{A^T \theta} P e^{A \theta} J - P < 0
\]

holds for all \(\theta \in [T_{\text{min}}, T_{\text{max}}]\).
Theorem

Let us consider an impulsive system \((A, J)\). Then, the following statements are equivalent:

(a) There exists a matrix \(P \in \mathbb{S}_+^n\) such that the LMI

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\]

holds for all \(\theta \in [T_{min}, T_{max}]\).

(b) There exist a differentiable matrix function \(R : [0, T_{max}] \mapsto \mathbb{S}^n, R(0) > 0, \) and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T R(\tau) + R(\tau) A - \dot{R}(\tau) \preceq 0
\]

and

\[
J^T R(\theta) J - R(0) + \varepsilon I \preceq 0
\]

hold for all \(\tau \in [0, T_{max}]\) and all \(\theta \in [T_{min}, T_{max}]\).
Theorem

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(a) There exists a matrix \(P \in \mathbb{S}_+^n\) such that the LMI

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holds for all \(\theta \in [T_{\text{min}}, T_{\text{max}}]\).

(b) There exist a differentiable matrix function \(R : [0, T_{\text{max}}] \mapsto \mathbb{S}^n\), \(R(0) > 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

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\]  

and

\[
J^T R(\theta) J - R(0) + \varepsilon I \preceq 0
\]

hold for all \(\tau \in [0, T_{\text{max}}]\) and all \(\theta \in [T_{\text{min}}, T_{\text{max}}]\).

Moreover, when one of the above statements holds, then the aperiodic impulsive system with ranged dwell-time \(T_k \in [T_{\text{min}}, T_{\text{max}}]\) is asymptotically stable.
Theorem (Minimum Dwell-Time)

Let us consider an impulsive system \( (A, J) \). Then, the following statements are equivalent:

(a) There exists a matrix \( P \in \mathbb{S}_+^n \) such that the LMIs

\[
A^T P + PA < 0 \quad \text{and} \quad J^T e^{A^T \bar{T}} P e^{A \bar{T}} J - P < 0
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hold.
Theorem (Minimum Dwell-Time)

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hold.

(b) There exist a differentiable matrix function \(R : [0, \bar{T}] \mapsto \mathbb{S}^n\), \(R(0) > 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T R(0) + R(0) A < 0
\]

\[
A^T R(\tau) + R(\tau) A - \dot{R}(\tau) \leq 0 \quad \text{and} \quad J^T R(\bar{T}) J - R(0) + \varepsilon I \leq 0
\]

hold for all \(\tau \in [0, \bar{T}]\).
Convex conditions for minimum dwell-time

Theorem (Minimum Dwell-Time)

Let us consider an impulsive system $(A, J)$. Then, the following statements are equivalent:

(a) There exists a matrix $P \in \mathbb{S}_n^+$ such that the LMIs

\[ A^T P + P A < 0 \quad \text{and} \quad J^T e^{A^T \bar{T}} P e^{A \bar{T}} J - P < 0 \]

hold.

(b) There exist a differentiable matrix function $R : [0, \bar{T}] \rightarrow \mathbb{S}_n$, $R(0) \succ 0$, and a scalar $\varepsilon > 0$ such that the LMIs

\[ A^T R(0) + R(0) A < 0 \]
\[ A^T R(\tau) + R(\tau) A - \dot{R}(\tau) \leq 0 \quad \text{and} \quad J^T R(\bar{T}) J - R(0) + \varepsilon I \leq 0 \]

hold for all $\tau \in [0, \bar{T}]$.

Moreover, when one of the above statements holds, the impulsive system is asymptotically stable under minimum dwell-time $\bar{T}$, i.e. for any sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $T_k \geq \bar{T}$. 
Pros and cons

Benefits

- Convex in the matrices of the system → robustness analysis possible
**Pros and cons**

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- Convex in the matrices of the system $\rightarrow$ robustness analysis possible
- Convex in the matrices of the system $\rightarrow$ control design possible
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- Convex in the matrices of the system $\rightarrow$ robustness analysis possible
- Convex in the matrices of the system $\rightarrow$ control design possible
- Applicable to systems with time-varying matrices and to nonlinear (polynomial) systems
Pros and cons

Benefits

- Convex in the matrices of the system → robustness analysis possible
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- Applicable to systems with time-varying matrices and to nonlinear (polynomial) systems

Drawbacks

- Infinite-dimensional LMI problems
Pros and cons

Benefits

- Convex in the matrices of the system → robustness analysis possible
- Convex in the matrices of the system → control design possible
- Applicable to systems with time-varying matrices and to nonlinear (polynomial) systems

Drawbacks

- Infinite-dimensional LMI problems
- Needs relaxation (piecewise linear approximation or polynomial functions (SOS))
Let $R(\tau)$ and $\Gamma(\tau)$ be polynomials of order $2d$ and assume that the following conditions hold:

- $R(0) - \varepsilon I \succeq 0$
- $A^T R(0) + R(0) A + \varepsilon I \preceq 0$
- $\Gamma(\tau)$ is a SOS matrix, i.e. there exists $M(\tau)$ such that $\Gamma(\tau) = M(\tau)^T M(\tau)$.
- $-A^T R(\tau) - R(\tau) A + \dot{R}(\tau) - \Gamma(\tau) \tau (T - \tau)$ is a SOS matrix.
- $J^T R(\bar{T}) J - R(0) + \varepsilon I \preceq 0$.

Then, the impulsive system is asymptotically stable under minimum dwell-time $\bar{T}$, i.e. for any sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $T_k \geq \bar{T}$. 
Example 1 - Range dwell-time

Let us consider the system\(^1\)

\[
A = \begin{bmatrix}
-1 & 0.1 \\
0 & 1.2
\end{bmatrix}, \quad J = \begin{bmatrix}
1.2 & 0 \\
0 & 0.5
\end{bmatrix}. \tag{10}
\]

Example 1 - Range dwell-time

Let us consider the system

$$A = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \quad J = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}. \quad (10)$$

<table>
<thead>
<tr>
<th>$d_R$</th>
<th>$T_{min}$</th>
<th>$T_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>0.4998</td>
</tr>
<tr>
<td>4</td>
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<td>0.5768</td>
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<td>6</td>
<td>0.1824</td>
<td>0.5776</td>
</tr>
<tr>
<td>Periodic case</td>
<td>0.1824</td>
<td>0.5776</td>
</tr>
</tbody>
</table>

- Finds the theoretical bounds
- Also holds in the aperiodic case

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Example 2 - Minimum dwell-time

Let us consider the system

\[
A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.
\] (11)

---

Example 2 - Minimum dwell-time

Let us consider the system \(^1\)

\[
A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.
\]  

<table>
<thead>
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<th>(d_R)</th>
<th>(T_{min})</th>
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<td>Proposed approach</td>
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<tr>
<td>Exponential LMI</td>
<td>–</td>
</tr>
<tr>
<td>Periodic case</td>
<td>–</td>
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</table>

- Non-conservative dwell-time result

Stabilization of impulsive systems
Stabilization problem

System

\begin{align}
\dot{x}(t) &= Ax(t) + B_c u_c(t), \ t \neq t_k \\
x(t) &= Jx(t^-) + B_d u_d(k), \ t = t_k
\end{align}

(12)

where $u_c : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m_c}$ and $u_d : \mathbb{N} \to \mathbb{R}^{m_d}$ are the control inputs.
Stabilization problem

System

\[\begin{align*}
\dot{x}(t) &= Ax(t) + B_c u_c(t), \quad t \neq t_k \\
x(t) &= Jx(t^-) + B_d u_d(k), \quad t = t_k
\end{align*}\]  \hfill (12)

where \( u_c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c} \) and \( u_d : \mathbb{N} \rightarrow \mathbb{R}^{m_d} \) are the control inputs.

Control law

We consider the following class of control-laws:

\[\begin{align*}
u_c(t_k + \tau) &= K_c(\tau)x(t_k + \tau), \quad \tau \in [0, T_k), \\
u_d(k) &= K_d x(t_k^-)
\end{align*}\]  \hfill (13)
Stabilization problem

System

\begin{align}
\dot{x}(t) &= Ax(t) + B_c u_c(t), \quad t \neq t_k \\
x(t) &= Jx(t^-) + B_d u_d(k), \quad t = t_k
\end{align}

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where \(u_c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c}\) and \(u_d : \mathbb{N} \rightarrow \mathbb{R}^{m_d}\) are the control inputs.

Control law

We consider the following class of control-laws:

\begin{align}
\begin{aligned}
\quad u_c(t_k + \tau) &= K_c(\tau)x(t_k + \tau), \quad \tau \in [0, T_k), \\
\quad u_d(k) &= K_d x(t_k^-)
\end{aligned}
\end{align}

(13)

Minimum dwell-time case

\[
K_c(\tau) = \begin{cases} 
\tilde{K}_c(\tau) & \text{if } \tau \in [0, \bar{T}) \\
\tilde{K}_c(\bar{T}) & \text{if } \tau \in [\bar{T}, T_k)
\end{cases}
\]

(14)

where \(T_k \geq \bar{T}, k \in \mathbb{N}\) and \(\tilde{K}_c(\tau)\) is some matrix function to be determined.
Theorem (Minimum dwell-time)

Assume that there exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(\bar{T}) \succ 0$, a matrix function $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n}$, a matrix $U_d \in \mathbb{R}^{m_d \times n}$ and a scalar $\varepsilon > 0$ such that the LMIs

$$\text{Sym}[AS(\bar{T}) + B_c U_c(\bar{T})] \prec 0,$$

$$\text{Sym}[AS(\tau) + B_c U_c(\tau)] + \dot{S}(\tau) \preceq 0$$

and

$$\begin{bmatrix}
-S(0) + \varepsilon I & JS(\bar{T}) + B_d U_d \\
* & -S(\bar{T})
\end{bmatrix} \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$. Then, the closed-loop system is asymptotically stable with minimum dwell-time $\bar{T}$ and suitable controller gains are retrieved using

$$\tilde{K}_c(\tau) = U_c(\tau)S(\tau)^{-1} \quad \text{and} \quad K_d = U_dS(\bar{T})^{-1}. \quad (18)$$
Theorem (Range dwell-time)

Assume that there exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(0) \succ 0$, a matrix function $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n}$, a matrix $U_d \in \mathbb{R}^{m_d \times n}$ and a scalar $\varepsilon > 0$ such that the LMIs

$$\text{Sym}[AS(\tau) + B_c U_c(\tau)] + \dot{S}(\tau) \preceq 0$$

(19)

and

$$\begin{bmatrix}
-S(\theta) + \varepsilon I & JS(0) + B_d U_d \\
* & -S(0)
\end{bmatrix} \preceq 0$$

(20)

hold for all $\tau \in [0, T_{\text{max}}]$ and all $\theta \in [T_{\text{min}}, T_{\text{max}}]$. Then, the closed-loop system is asymptotically stable with range dwell-time $[T_{\text{min}}, T_{\text{max}}]$ and suitable controller gains are retrieved using

$$\hat{K}_c(\tau) = U_c(\tau)S(\tau)^{-1} \quad \text{and} \quad K_d = U_dS(0)^{-1}.$$
Example

Let us consider the system with matrices

\[
A = \begin{bmatrix}
1 & 0 \\
1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}
1 & 1 \\
1 & 3
\end{bmatrix}
\] (22)

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \tilde{T} = 0.1 \).
Let us consider the system with matrices

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}
\] (22)

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \bar{T} = 0.1 \).
- We obtain

\[
\tilde{K}_c(\tau) = \frac{1}{\text{den}(\tau)} \begin{bmatrix} 1.4750481 + 3.2714889\tau - 41.011914\tau^2 \\ 3.9063911 - 1.6733059\tau - 37.472443\tau^2 \end{bmatrix}^T
\]

where \( \text{den}(\tau) = -0.19767438 + 0.78454217\tau + 7.6562219\tau^2 \).
Let us consider the system with matrices

\[ A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \]  \quad (22)

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \bar{T} = 0.1 \).
“Applications”
Switched systems

A linear switched system is a system of the form

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad x_0 \in \mathbb{R}^n, \quad t \geq 0 \tag{23} \]

where \( \sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \ldots, N\} \) is the switching signal.
Switched systems

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(23)

where $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \ldots, N\}$ is the switching signal.

Impulsive system formulation

$$\begin{align*}
\dot{\tilde{x}}(t) &= \text{diag}\{A_1, \ldots, A_N\}\tilde{x}(t), \ \tilde{x}_0 = e_1 \otimes x_0, \ t \geq 0 \\
\tilde{x}(t) &= (e_j^T e_i) \otimes I_n, \ i, j = 1, \ldots, N, \ i \neq j
\end{align*}$$

(24)

- Reset system with multiple reset maps
- The method can be applied and leads to the conditions discussed in\(^1\) when using block-diagonal Lyapunov functions

---


\(^2\) C. Briat. Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time, *Systems & Control Letters*, 2015a
Convex conditions for minimum dwell-time

Theorem (Minimum Dwell-Time)

Let us consider a switched system with matrices \( \{A_1, \ldots, A_N\} \). Then, the following statements are equivalent:

(a) There exist matrices \( P_i \in \mathbb{S}_+^n, i = 1, \ldots, N \), such that the LMIs

\[
A_i^T P_i + P_i A_i < 0 \quad \text{and} \quad e^{A_i^T \bar{T}} P_i e^{A_i \bar{T}} - P_j < 0
\]

hold for all \( i, j = 1, \ldots, N, i \neq j \).
Theorem (Minimum Dwell-Time)

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\]

hold for all \( i, j = 1, \ldots, N, i \neq j \).

(b) There exist differentiable matrix functions \( R_i : [0, \bar{T}] \mapsto \mathbb{S}^n \), \( R_i(0) > 0 \), \( i = 1, \ldots, N \), and a scalar \( \varepsilon > 0 \) such that the LMIs

\[
A_i^T R_i(0) + R_i(0) A_i < 0
\]

\[
A_i^T R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) \preceq 0 \quad \text{and} \quad R_i(\bar{T}) - R_j(0) + \varepsilon I \preceq 0
\]

hold for all \( \tau \in [0, \bar{T}] \) and for all \( i, j = 1, \ldots, N, i \neq j \).
Theorem (Minimum Dwell-Time)

Let us consider a switched system with matrices \( \{A_1, \ldots, A_N\} \). Then, the following statements are equivalent:

(a) There exist matrices \( P_i \in S^n_{>0}, i = 1, \ldots, N \), such that the LMIs

\[
A_i^T P_i + P_i A_i \prec 0 \quad \text{and} \quad e^{A_i T} P_i e^{A_i \bar{T}} - P_j \prec 0
\]

hold for all \( i, j = 1, \ldots, N, i \neq j \).

(b) There exist differentiable matrix functions \( R_i : [0, \bar{T}] \to S^n, R_i(0) \succ 0 \), \( i = 1, \ldots, N \), and a scalar \( \varepsilon > 0 \) such that the LMIs

\[
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\]

\[
A_i^T R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) \preceq 0 \quad \text{and} \quad R_i(\bar{T}) - R_j(0) + \varepsilon I \preceq 0
\]

hold for all \( \tau \in [0, \bar{T}] \) and for all \( i, j = 1, \ldots, N, i \neq j \).

Moreover, when one of the above statements holds, the switched system is asymptotically stable under minimum dwell-time \( \bar{T} \), i.e. for any sequence \( \{t_k\}_{k \in \mathbb{N}} \) such that \( T_k \geq \bar{T} \).
System
Let us consider now the continuous-time system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(25)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state of the system and the control input, respectively.
System

Let us consider now the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t)$$  \hspace{1cm} (25)

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system and the control input, respectively.

Controller

The control input is assumed to be computed from a sampled-data state-feedback control law given by

$$u(t) = K_1 x(t_k) + K_2 u(t_{k-1}), \quad t \in [t_k, t_{k+1})$$ \hspace{1cm} (26)

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ are the control gains to be determined.
Sampled-data systems

System
Let us consider now the continuous-time system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(25)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state of the system and the control input, respectively.

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The control input is assumed to be computed from a sampled-data state-feedback control law given by

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(26)

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times m} \) are the control gains to be determined.

Objectives
Find a control law of the form (26) such that the closed-loop system is robustly stable for all sampling-periods in the range \([T_{min}, T_{max}]\).
Any sampled-data system can be equivalently reformulated as an impulsive system:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{z}(t)
\end{bmatrix} = \begin{bmatrix}
A & B \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x(t) \\
z(t)
\end{bmatrix}, \quad t \neq t_k
\]

\[
\begin{bmatrix}
x(t) \\
z(t)
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
K_1 & K_2
\end{bmatrix} \begin{bmatrix}
x(t^-) \\
z(t^-)
\end{bmatrix}, \quad t = t_k
\]

where \( z(t) = u(t_k), \quad t \in [t_k, t_{k+1}) \).

Let \( \bar{J} = J_0 + B_0 K \) where

\[
J_0 = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
0 \\
I
\end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix}
K_1 & K_2
\end{bmatrix}.
\]
Theorem (Aperiodic sampled-data systems)

The following statements are equivalent:

(a) There exists a control law of the form (26) that quadratically stabilizes the system (25) for any aperiodic sampling instant sequence \( \{t_k\} \) with dwell-time \( T_k \in [T_{\text{min}}, T_{\text{max}}] \).
Theorem (Aperiodic sampled-data systems)

The following statements are equivalent:

(a) There exists a control law of the form (26) that quadratically stabilizes the system (25) for any aperiodic sampling instant sequence \( \{t_k\} \) with dwell-time \( T_k \in [T_{\text{min}}, T_{\text{max}}] \).

(b) There exist a differentiable matrix function \( R : [0, T_{\text{max}}] \mapsto S^{n+m}, S(0) \succ 0 \), a matrix \( Y \in \mathbb{R}^{m \times (n+m)} \) and a scalar \( \varepsilon > 0 \) such that the conditions

\[
\bar{A}(\tau)S(\tau) + S(\tau)\bar{A}(\tau)^T + \dot{S}(\tau) \preceq 0
\]

and

\[
\begin{bmatrix}
-S(\theta) + \varepsilon I & J_0S(0) + B_0Y \\
* & -S(0)
\end{bmatrix} \preceq 0
\]

hold for all \( \tau \in [0, T_{\text{max}}] \) and all \( \theta \in [T_{\text{min}}, T_{\text{max}}] \).

Moreover, when this statement holds, a suitable stabilizing control gain can be obtained using the expression \( K = YS(0)^{-1} \).
Example 1

Let us consider the sampled-data system (25) with matrices

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \] (31)
Example 1

Let us consider the sampled-data system (25) with matrices

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.
\]  

(31)

- Fixed control law: \( K_1 = \begin{bmatrix} -3.75 \\ -11.5 \end{bmatrix} \) and \( K_2 = 0 \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>( T_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed result</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>(Fridman et al., 2004)</td>
<td>–</td>
</tr>
<tr>
<td>(Naghshtabrizi et al., 2008)</td>
<td>–</td>
</tr>
<tr>
<td>(Fridman, 2010)</td>
<td>–</td>
</tr>
<tr>
<td>(Liu et al., 2010)</td>
<td>–</td>
</tr>
<tr>
<td>(Seuret, 2012)</td>
<td>–</td>
</tr>
<tr>
<td>(Seuret and Peet, 2013)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>
Example 1

Let us consider the sampled-data system (25) with matrices

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \]  

(31)

- Designed control law for some given \([T_{min}, T_{max}]\).

<table>
<thead>
<tr>
<th>(T_{min})</th>
<th>(T_{max})</th>
<th>(K_1)</th>
<th>(K_2)</th>
<th>(d_R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>10</td>
<td>-0.1145</td>
<td>-0.8088</td>
<td>-0.0024</td>
</tr>
<tr>
<td>0.001</td>
<td>50</td>
<td>-0.0202</td>
<td>-0.1560</td>
<td>-0.0030</td>
</tr>
<tr>
<td>0.001</td>
<td>10</td>
<td>-0.0310</td>
<td>-0.3222</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>50</td>
<td>-0.0259</td>
<td>-0.2726</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 2

- Let us consider the following sampled-data system (25) with matrices

\[
A = \begin{bmatrix}
0 & 1 \\
-2 & 0.1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]  

(32)

- Let \( K_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \) and \( K_2 = 0 \).

<table>
<thead>
<tr>
<th>( d_R )</th>
<th>Proposed result</th>
<th>(Seuret, 2012)</th>
<th>(Seuret and Peet, 2013)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{min} )</td>
<td>4 \quad 6</td>
<td>0.4 \quad 0.4</td>
<td>1.6316 \quad 1.8270</td>
</tr>
<tr>
<td>( T_{max} )</td>
<td>1.251 \quad 1.828</td>
<td>1.820 \quad 1.828</td>
<td></td>
</tr>
</tbody>
</table>
Concluding remarks
Concluding statements

- Robust stability under minimum and range dwell-time
- Robust stabilization by state feedback possible
- Easily extensible to the case of homogeneous Lyapunov functions (necessity?)

Possible extensions

- Switched systems\(^1\), stochastic systems\(^2\), LPV systems (PC parameters\(^3\))
- Dynamic output feedback? There is hope… (according to some trustable people)
- Nonlinear systems (low hanging fruit, just got a paper to review yesterday where the authors seem to do that)
- Similar ideas for delay systems (I start to have some ideas on how to do…)

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\(^1\) C. Briat. Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time, *Systems & Control Letters*, 2015a

\(^2\) C. Briat. Stability analysis and stabilization of stochastic linear impulsive systems – applications to sampled-data systems, *submitted to Automatica*, 2015c

\(^3\) C. Briat. Stability analysis and control of LPV systems with piecewise constant parameters, *Systems & Control Letters*, 2015b
Thank you for your Attention